Non-Abelian expansion of congruence KMS complexes

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Abstract

Coboundary expansion with non-Abelian coefficients is a strong version of high-dimensional expansion for simplicial complexes. One motivation for studying this notion is that it was recently shown to have deep connections to problems in theoretical computer science. However, very few examples of families of simplicial complexes with this type of expansion are known. Namely, prior to our work, the only known examples were quotients of symplectic buildings and a slight variation of the Kaufman-Oppenheim coset complexes construction associated with $SL_n(\mathbb{F}_p[t])$.

In this paper, we show that the Grave de Peralta and Valentiner-Branth constructions of KMS complexes have coboundary expansion with non-Abelian coefficients when it is performed with respect to congruence subgroups of Chevalley groups of classical type, i.e., of type A_n, B_n, C_n and D_n. This gives four new sources of examples to this expansion phenomenon, thus significantly enriching our list of constructions.

1 Introduction

Given a simplicial complex X, the cohomology of X is usually defined with coefficients in an Abelian group. However, when one considers the low dimensional cohomology, the definition can be generalized to computing the 0-th and first cohomologies of X with respect to any coefficient group Λ where Λ need not be Abelian. Given a group Λ, 1-coboundary expansion of X is a quantization of the vanishing of $H^1(X, \Lambda)$ (see exact definition in Section 2.1.3 below). This notion of coboundary expansion with non-Abelian coefficients has received much attention lately due to its deep connections with theoretical computer science [BM24, DD24b, BLM24, DDL24, BMVY25] where the main focus is choosing the coefficient group Λ to be a finite symmetric group.

As in the case of expander graphs, the focus is not on a single simplicial complex, but rather on a family of complexes. Given a group Λ, a family of simplicial complexes $\{X^{(s)}\}$ is said to be a family of 1-coboundary expanders with Λ coefficients if it has uniformly bounded degree, the size of $X^{(s)}$ grows with s to infinity and all the complexes $X^{(s)}$ have a uniform bound on their Λ 1-coboundary expansion (see exact definition in Section 2.1.3 below).

Prior to our work, for a fixed finite symmetric group Λ, only two constructions of families of 1-coboundary expanders with Λ coefficients were known: A construction coming from quotients of symplectic buildings (see [BLM24, DDL24]) and a construction coming from a coset complexes over the group $\mathrm{SL}_{n+1}(\mathbb{F}_p[t])$ (see [KOW24]). In this work, we expand the list of known constructions and show how the constructions of certain types of KMS complexes that were introduced in [GdPVB25] give rise to 1-coboundary expanders with Λ coefficients.

In [GdPVB25], it was shown how to utilize KMS groups to construct high-dimensional spectral expanders. As a specific instantiation of their general method, the authors of [GdPVB25] showed how to construct a family of special KMS complexes, which we will call here *congruence KMS complexes* (this is our terminology and it did not appear in [GdPVB25]). The construction of congruence KMS complexes in [GdPVB25] can be roughly described as follows: Given a spherical root system $\mathring{\Phi}$, a finite field K and an irreducible polynomial $f \in K[t]$, the construction in [GdPVB25] outputs a finite simplicial complex $X = X(\mathring{\Phi}, K, f)$ that is a coset complex (see exact details in Section 2.2.4 and a first example in Section 1.3 below). This construction is best behaved when $\mathrm{char}(K) \neq 2$, hence we will assume is for the rest of this paper, although the assumption is only necessary for type B_n and C_n and even the characteristic two case could be handled with a bit more technical work, see [GdPVB25, Remark 5.10].

With this notation, our main result can be stated as follows:

Theorem 1.1 (See Corollary 5.3 for a more general formulation)**.** *Let Λ be a group, $n \geq 3$ an integer and $\mathring{\Phi}$ a spherical root system of type A_n, B_n, C_n or D_n. Also, let K be a finite field of characteristic $p \neq 2$ such that $|K|$ is sufficiently large with respect to n and that Λ does not have an element of order p (e.g., if Λ is finite and $p > |\Lambda|$). Fix $f_s \in K[t]$ to be a sequence of irreducible polynomials such that $\lim_s \deg(f_s) = \infty$ and denote $X^{(s)} = X(\mathring{\Phi}, K, f_s)$. Then the family $\{X^{(s)}\}_s$ is a family of 1-coboundary expanders with Λ coefficients.*

This Theorem is deduced from two separate results that are independent from each other:

- A vanishing result for the first non-Abelian cohomology for congruence KMS complexes.

- A 1-cosystolic expansion with non-Abelian coefficients for congruence KMS complexes.

1.1 Vanishing of the first cohomology

For congruence KMS complexes we prove the following vanishing result:

Theorem 1.2 (See Corollary 3.8)**.** *Let Λ be a finite group, $n \geq 3$ an integer and $\mathring{\Phi}$ a spherical root system. Also, let K be a finite field of characteristic $p \neq 2$ such that $|K| \geq 5$ and such that Λ does not have an element of order p (e.g., if Λ is finite and $p > |\Lambda|$). For any irreducible polynomial $f \in K[t]$ of degree at least 2, it holds that $H^1(X(\mathring{\Phi}, K, f), \Lambda) = 0$.*

It is worthwhile to compare our proof of the vanishing of cohomology with recent results in [BLM24, DDL24] and in [KOW24]. For a fixed finite group Λ all the works ([BLM24, DDL24,

KOW24] and ours) use the same basic methodology to construct a sequence of finite simplicial complexes with vanishing first Λ-cohomology. We all start with a universal "algebraic" simplicial complex that has trivial 1-cohomology with respect to Λ, and then passes to a family of finite complexes by taking quotients via congruence subgroups, which also have trivial 1-cohomology with Λ-coefficients.

Applying this methodology yields different difficulties that are dependent on the specific construction:

- In [BLM24, DDL24], the initial complex is a contractible symplectic $\widetilde{C}_n$-building, which has trivial 1-cohomology for any group. The difficulty lies in proving that the congruence subgroups exhibit trivial 1-cohomology with respect to Λ, requiring deep results from group theory, such as the congruence subgroup property and the strong approximation theorem.

- In [KOW24], the quotients are taken over a coset complex over $\mathrm{SL}_{n+1}(\mathbb{F}_p[t])$. For $n \geq 3$, there is an explicit description of the congruence subgroups, which follows from the fact that $\mathrm{SL}_{n+1}(\mathbb{F}_p[t])$ has the same presentation as a Steinberg group. This explicit description gives rise to a straightforward argument for proving the vanishing of 1-cohomology with Λ-coefficients given that Λ has no element of order p (e.g., when Λ is finite and $p > |\Lambda|$). Thus, most of the work in [KOW24] focuses on proving that the coset complex over $\mathrm{SL}_{n+1}(\mathbb{F}_p[t])$ has trivial 1-cohomology with respect to Λ.

- At least with respect to proving vanishing of cohomology, the congruence KMS complexes enjoy the best of both worlds: First, the initial complex from which we take quotients is simply connected and thus has trivial 1-cohomology for any group. Second, showing the vanishing of 1-cohomology for congruence subgroups with Λ-coefficients can be done via the presentation theory of the Chevalley groups (the argument shares some similarity with that of [KOW24]). The takeaway is that the proof of vanishing of the first cohomology with Λ coefficients for congruence KMS complexes is perhaps the simplest in all known examples.

1.2 Non-Abelian cosystolic expansion

Cosystolic expansion is a weaker variant of coboundary expansion. There is a known local to global machinery for proving cosystolic expansion developed by [KKL14, EK24] for Abelian coefficients and [KM21, DD24a] for non-Abelian coefficients. Namely, in order to show cosystolic expansion for a finite simplicial complex, it is enough to show local spectral expansion and coboundary expansion of the links. Local spectral expansion for congruence KMS complexes was shown in [GdPVB25] so we are left to prove non-Abelian coboundary expansion of the links.

In [OVB25], we already showed that links of congruence KMS complexes have Abelian coboundary expansion in all relevant dimensions. Here, we generalize our method from [OVB25] to show that the links of congruence KMS complexes have 1-coboundary expansion with respect to any group Λ:

Theorem 1.3 (See Theorem 5.1 for a more general formulation)**.** *Let $n \geq 3$ be an integer and $\mathring{\Phi}$ a spherical root system of type A_n, B_n, C_n or D_n. Also, let K be a finite field of characteristic > 2 such that $|K|$ is sufficiently large with respect to n. There is a constant $\varepsilon' > 0$ such that any group Λ, any irreducible polynomial $f \in K[t]$ of degree at least 2 and any vertex v in $X(\mathring{\Phi}, K, f_s)$, the Λ 1-coboundary expansion of $\mathrm{lk}(v)$ is $\geq \varepsilon'$.*

Combining it with the results of [DD24a] yields that congruence KMS complexes are 1-cosystolic expanders:

Theorem 1.4 (Corollary 5.2). *Let $n \geq 3$ be an integer and $\mathring{\Phi}$ a spherical root system of type A_n, B_n, C_n or D_n. Also, let K be a finite field such that $|K|$ is sufficiently large with respect to n and $\mathrm{char}(K) \neq 2$. There is a constant $\varepsilon > 0$ such that for any group Λ and any irreducible polynomial $f \in K[t]$ of degree at least 2, the Λ 1-cosystolic expansion of the complex $X(\mathring{\Phi}, K, f_s)$ is $\geq \varepsilon$.*

Remark 1.5. Our results regarding Λ 1-cosystolic expansion are stated above for congruence KMS complexes, but it is worth noting that they hold for a larger class of KMS complexes, namely of n-classical KMS complexes (see Definition 2.3). As noted above, the more general results are stated in Theorem 5.1 and Corollary 5.2 below.

1.3 A_n congruence KMS complexes

Our result stated above regards congruence KMS complexes of type A_n, B_n, C_n and D_n. In order to give a concrete example, here we will explicitly describe the congruence KMS complexes of type A_n and state our result for these complexes.

All congruence KMS complexes that we consider are coset complexes. More details on coset complexes can be found for example in [KO23], and we will only give a brief overview.

Definition 1.6. Let G be a group and $\mathcal{H} = (H_0, \ldots, H_n)$ be a family of subgroups of G. Then the coset complex $\mathcal{CC}(G; (H_0, \ldots, H_n))$ is defined to be the simplicial complex with

- vertex set $\bigsqcup_{i=0}^n G/H_i$

- a set of vertices $\{g_1 H_{i_1}, \ldots, g_k H_{i_k}\}$ forms a $(k-1)$-simplex in $\mathcal{CC}(G; \mathcal{H})$ if and only if $\bigcap_{j=1}^k g_j H_{i_j} \neq \emptyset$.

Note that the set of vertices is partitioned into $n+1$ subsets G/H_i for $i = 0, \ldots, n$ such that for any simplex $\sigma \in \mathcal{CC}(G; \mathcal{H})$ we have $|\sigma \cap G/H_i| \leq 1$ for all i.

With this notation, we will define a coset complex of a subgroup of $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ where $n \geq 2$ and q is an odd prime power such that $q \geq 5$. For $1 \leq i, j \leq n+1, i \neq j$ and $f \in \mathbb{F}_q[t]$, we denote $e_{i,j}(f) \in \mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ to be the matrix with 1's along the main diagonal, f in the (i,j)-th entry and 0's in all other entries. Let

$$B = \{e_{i,i+1}(1); i = 1, \ldots n\} \cup \{e_{n+1,1}(t)\}$$

and let $\mathcal{U}(\mathbb{F}_q) < \mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ be the subgroup generated by B, i.e.

$$\mathcal{U}(\mathbb{F}_q) = \langle e_{1,2}(1), e_{2,3}(1), \ldots, e_{n,n+1}(1), e_{n+1,1}(t) \rangle.$$

We define subgroups $H_0, \ldots, H_n < \mathcal{U}(\mathbb{F}_q)$ as

$$H_0 = \langle e_{1,2}(1), e_{2,3}(1), \ldots, e_{n,n+1}(1) \rangle = \langle B \setminus \{e_{n+1,1}(t)\} \rangle,$$
$$H_1 = \langle e_{2,3}(1), \ldots, e_{n,n+1}(1), e_{n+1,1}(t) \rangle = \langle B \setminus \{e_{1,2}(1)\} \rangle,$$
$$\vdots$$
$$H_i = \langle B \setminus \{e_{i,i+1}(1)\} \rangle,$$
$$\vdots$$
$$H_n = \langle e_{1,2}(1), e_{2,3}(1), \ldots, e_{n-1,n}(1), e_{n+1,1}(t) \rangle,$$

and denote $\mathcal{H} = (H_0, \ldots, H_n)$.

Let $f \in \mathbb{F}_q[t]$ be an irreducible polynomial of degree $s > 1$. Denote $\phi_f : \mathrm{SL}_{n+1}(\mathbb{F}_q[t]) \to \mathrm{SL}_{n+1}(\mathbb{F}_q[t]/(f)) \cong \mathrm{SL}_{n+1}(\mathbb{F}_{q^s})$ defined as follows: For every matrix $A \in \mathrm{SL}_{n+1}(\mathbb{F}_q[t]), A = (A_{i,j})_{i,j=1}^n$ where $A_{i,j} \in \mathbb{F}_q[t]$ define

$$\phi_f(A) = (A_{i,j} + (f))_{i,j=1}^n.$$

Restricting ϕ_f to $\mathcal{U}(\mathbb{F}_q)$, we get $G_f = \phi_f(\mathcal{U}(\mathbb{F}_q))$ which is a finite quotient of $\mathcal{U}(\mathbb{F}_q)$. It is proven in [GdPVB25], that ϕ_f is injective on $H_0, \ldots, H_n$. The congurence KMS complex of type A_n over K with respect to f is defined to be the coset complex $X_f = \mathcal{CC}(G_f, \mathcal{H}_f)$, where $\mathcal{H}_f = (\phi_f(H_0), \ldots, \phi_f(H_n))$.

Let f_s be a sequence of irreducible polynomials of degrees ≥ 2 such that $\deg(f_s) \to \infty$. Define $X^{(s)} = \mathcal{CC}(G_{f_s}, \mathcal{H}_{f_s})$. In [GdPVB25] (see also Section 2.2 below) it was proven that for every large enough q with respect to n, the sequence $\{X^{(s)}\}_s$ is a family of uniformly bounded degree pure n-dimensional simplicial complexes that are $\frac{1}{\sqrt{q}-(n-1)}$-local spectral expanders.

All the links of all the vertices in all the $X^{(s)}$'s are isomorphic to the coset complex $\mathcal{CC}(H_0; (H_0 \cap H_1, ..., H_0 \cap H_n))$ (where $H_0, ..., H_n$ are as above). The coset complex $\mathcal{CC}(H_0; (H_0 \cap H_1, ..., H_0 \cap H_n))$ is isomorphic to the opposite complex inside the spherical building of type A_n over the field $\mathbb{F}_q$.

With respect to the family $\{X^{(s)}\}_s$ defined above, our main result is as follows:

Theorem 1.7. *Let Λ be a finite group, $n \geq 3$ be an integer, $q = p^m$ where $p > 2$ is prime. Also let $\{X^{(s)}\}_s$ be the family of congruence KMS complexes constructed above as coset complexes over quotients of $\mathcal{U}(\mathbb{F}_q) < \mathrm{SL}_{n+1}(\mathbb{F}_q[t])$. If $q > 2^{n-1}$ and $p > |\Lambda|$, then the family $\{X^{(s)}\}_s$, is a family of 1-coboundary expanders with Λ coefficients.*

1.4 Organization

The paper is organized as follows. In Section 2, we first recall the relevant notions of high-dimensional expansion in Section 2.1 and then introduce Chevalley groups, KMS groups and the associated complexes in Section 2.2. In Section 3, we prove vanishing of the first cohomology of the congruence KMS complexes with respect to many coefficient groups. Section 4 is devoted to non-Abelian cone functions, first introducing the concept in Section 4.1, then showing how to construct cone functions for joins and by adding certain vertices. In Section 5, we show the cosystolic expansion of classical KMS complexes in the non-Abelian setting. Appendix A provides further details on buildings, their opposite complexes and the relation to KMS complexes. Appendix B, Appendix C and Appendix D provide further details on the proof of non-Abelian coboundary expansion of the opposite complexes of type A_n, C_n and D_n, respectively.

Acknowledgements

2 Preliminaries

In the first part of this section, we recall the definition and some properties of high-dimensional expanders, following the work of [KO23] and [DM22] among others. In the second part, we recall the construction of the KMS complexes from [GdPVB25] and [OVB25], and recall some of their properties.

2.1 High-dimensional expanders

2.1.1 Weighted simplicial complexes

Let X be a finite n-dimensional simplicial complex. A simplicial complex X is called *pure n-dimensional* if every face in X is contained in some face of dimension n. The set of all k-faces of X is denoted $X(k)$, and we will be using the convention in which $X(-1) = \{\emptyset\}$. For every $0 \leq k \leq n$, the *k-skeleton of X* is the simplicial complex $\bigcup_{i=-1}^{k} X(i)$. We will say that X is *connected* if its 1-skeleton is a connected graph.

Given a pure n-dimensional simplicial complex X, the *weight function* $w : \bigcup_{k=-1}^{n} X(k) \to \mathbb{R}_+$ is defined to be

$$\forall \tau \in X(k), w(\tau) = \frac{|\{\sigma \in X(n) : \tau \subseteq \sigma\}|}{\binom{n+1}{k+1}|X(n)|}.$$

We note that w is normalized such that for each $-1 \leq k \leq n$, the function w can be thought of as a probability function on $X(k)$.

Given a finite pure n-dimensional complex X and $\tau \in X$, *the link of τ* is the subcomplex $\mathrm{lk}_X(\tau)$ defined as

$$\mathrm{lk}_X(\tau) = \{\sigma \in X : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in X\}.$$

We note that if $\tau \in X(k)$, then $\mathrm{lk}_X(\tau)$ is a pure $(n - k - 1)$-dimensional complex. We call $\mathrm{lk}_X(\tau)$ a *proper link*, if $1 \leq \dim(\mathrm{lk}_X(\tau)) \leq n - 1$.

Below, we will use the following abuse of notation and write $v \in X(0)$ instead of $\{v\} \in X(0)$.

Given a family of pure n-dimensional simplicial complexes $\{X^{(s)}\}_s$, we say that this family has *bounded degree* if

$$\sup_{s} \sup_{v \in X^{(s)}(0)} |\{\sigma \in X^{(s)}(n) : v \in \sigma\}| < \infty.$$

2.1.2 Local spectral expansion

Let X be a finite pure n-dimensional simplicial complex with $n \geq 1$ and w the weight function on X defined above. We define the stochastic matrix of the random walk on (the 1-skeleton of) X to

be the matrix indexed by $X(0) \times X(0)$ and defined as

$$M(v, u) = \begin{cases} \frac{w(\{u,v\})}{\sum_{\{u',v\} \in X(1)} w(\{u',v\})} & \{v, u\} \in X(1) \\ 0 & \{v, u\} \notin X(1) \end{cases}.$$

For $\lambda < 1$, we say that X is a *(one-sided) λ-spectral expander* if X is connected and the second largest eigenvalue of M is $\leq \lambda$. We say that X is a *(one-sided) λ-local spectral expander* if for every $-1 \leq k \leq n - 2$ and every $\tau \in X(k)$, the simplicial complex $\mathrm{lk}_X(\tau)$ is a (one-sided) λ-spectral expander. We note that every (one-sided) λ-local spectral expander is a (one-sided) λ-spectral expander, since the link of the empty set is X itself.

2.1.3   Cohomological notations, cosystolic and coboundary expansion

Here we recall the basic definitions regarding 1-coboundary and 1-cosystolic expansion of a simplicial complex X with coefficients in a general group Λ. These definitions appeared in several works (e.g., [DM22]) and we claim no originality here.

Throughout, we denote X to be a finite pure n-dimensional simplicial complex. For every $-1 \leq k \leq n$, $X(k)$ denotes the set of k-dimensional simplices of X and $X_{\mathrm{ord}}(k)$ to be the set of ordered k-simplices.

Given a group Λ, we define the space of k-cochains for $k = -1, 0, 1$ with values in Λ by

$$\begin{aligned} C^{-1}(X, \Lambda) &= \{\phi : \{\emptyset\} \to \Lambda\}, \\ C^0(X, \Lambda) &= \{\phi : X(0) \to \Lambda\}, \\ C^1(X, \Lambda) &= \{\phi : X_{\mathrm{ord}}(1) \to \Lambda : \forall (u, v) \in X_{\mathrm{ord}}(1), \phi((u, v)) = (\phi((v, u)))^{-1}\}. \end{aligned}$$

We note that we do not assume that Λ is Abelian or finite.

For $k = -1, 0$, we define the coboundary maps $d_k : C^k(X, \Lambda) \to C^{k+1}(X, \Lambda)$ as follows: First, for every $\phi \in C^{-1}(X, \Lambda)$ and every $v \in X(0)$, we define $d_{-1}\phi(v) = \phi(\emptyset)$. Second, for every $\phi \in C^0(X, \Lambda)$ and every $(v_0, v_1) \in X_{\mathrm{ord}}(1)$, we define $d_0\phi((v_0, v_1)) = \phi(v_0)\phi(v_1)^{-1}$. Last, we also define the map $d_1 : C^1(X, \Lambda) \to \{\phi : X_{\mathrm{ord}}(2) \to \Lambda\}$ as follows: for every $\phi \in C^1(X, \Lambda)$ and every $(v_0, v_1, v_2) \in X_{\mathrm{ord}}(2)$, we define $d_1\phi((v_0, v_1, v_2)) = \phi((v_0, v_1))\phi((v_1, v_2))\phi((v_2, v_0))$.

We note that $d_k d_{k-1} \equiv e_\Lambda$ for $k = 0, 1$ and define cocycles and coboundaries by

$$\begin{aligned} Z^k(X, \Lambda) &= \{\phi \in C^0(X, \Lambda) : d_k\phi \equiv e_\Lambda\}, \\ B^k(X, \Lambda) &= d_{k-1}(C^{k-1}(X, \Lambda)). \end{aligned}$$

We note that $B^k(X, \Lambda) \subseteq Z^k(X, \Lambda)$ (as sets).

We further define the an action of $C^{k-1}(X, \Lambda)$ on $Z^k(X, \Lambda)$:

- An element $\psi \in C^{-1}(X, \Lambda)$ acts on $Z^0(X, \Lambda)$ by

$$\psi.\phi(v) = \psi(\emptyset)^{-1}\phi(v), \forall \phi \in Z^0(X, \Lambda), \forall v \in X(0).$$

- An element $\psi \in C^0(X, \Lambda)$ acts on $Z^1(X, \Lambda)$ by

$$\psi.\phi((v_0, v_1)) = \psi(v_0)\phi((v_0, v_1))\psi(v_1)^{-1}, \forall \phi \in Z^1(X, \Lambda), \forall (v_0, v_1) \in X_{\mathrm{ord}}(1).$$

We denote by $[\phi]$ the orbit of $\phi \in Z^k(X, \Lambda)$ under the action of $C^{k-1}(X, \Lambda)$ and define the reduced k-th cohomology to be

$$H^k(X, \Lambda) = \{[\phi] : \phi \in Z^k(X, \Lambda)\}.$$

We will say that the k-th cohomology is trivial if $H^k(X, \Lambda) = \{[\phi \equiv e_\Lambda]\}$ and note that the cohomology is trivial if and only if $B^k(X, \Lambda) = Z^k(X, \Lambda)$.

When Λ is an Abelian group, $Z^k(X, \Lambda), B^k(X, \Lambda)$ are Abelian groups and for $k = 0, 1$, the k-th reduced cohomology is defined by $H^k(X, \Lambda) = Z^k(X, \Lambda)/B^k(X, \Lambda)$. We leave it to the reader to verify that the definition in the Abelian setting coincides (as sets) with the definition in the general setting given above.

In order to define expansion, we need to define a norm. Let $w : \bigcup_{k=-1}^{n} X(k) \to \mathbb{R}_+$ be the weight function defined above. For $\phi \in C^0(X, \Lambda)$, we define $\text{supp}(\phi) \subseteq X(0)$ as

$$\text{supp}(\phi) = \{v \in X(0) : \phi(v) \neq e_\Lambda\}.$$

Also, for $\phi \in C^1(X, \Lambda)$, we define

$$\text{supp}(\phi) = \{\{u, v\} \in X(1) : \phi((u, v)) \neq e_\Lambda\},$$

and

$$\text{supp}(d_1\phi) = \{\{u, v, w\} \in X(2) : (d_1\phi)((u, v, w)) \neq e_\Lambda\}.$$

We note that for every permutation π on $\{0, 1, 2\}$ it holds that $d_1\phi((v_0, v_1, v_2)) = e_\Lambda$ if and only if $d_1\phi((v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)})) = e_\Lambda$ and thus $\text{supp}(d_1\phi)$ if well-defined.

With this notation, we define the following norms: for $\phi \in C^k(X, \Lambda)$, $k = 0, 1$, we define

$$\|\phi\| = \sum_{\tau \in \text{supp}(\phi)} w(\tau).$$

Also, for $\phi \in C^1(X, \Lambda)$, we define

$$\|d_1\phi\| = \sum_{\tau \in \text{supp}(d_1\phi)} w(\tau).$$

Last, for two maps $\phi, \psi \in C^k(X, \Lambda), k = 0, 1$, we define $\text{dist}(\phi, \psi)$ as follows: For $k = 0$, we define

$$\text{dist}(\phi, \psi) = w\left(\{v \in X(0) : \phi(v)(\psi(v))^{-1} \neq e_\Lambda\}\right).$$

For $k = 1$, we define

$$\text{dist}(\phi, \psi) = w\left(\{\{u, v\} \in X(1) : \phi((u, v))(\psi((u, v)))^{-1} \neq e_\Lambda\}\right).$$

Using the notations above, we will define coboundary and cosystolic expansion constants (generalizing the Cheeger constant) as follows: For $k = 0, 1$, define

$$h_{\text{cb}}^k(X, \Lambda) = \min_{\phi \in C^k(X, \Lambda) \setminus B^k(X, \Lambda)} \frac{\|d_k\phi\|}{\min_{\psi \in B^k(X, \Lambda)} \text{dist}(\phi, \psi)},$$

$$h_{\text{cs}}^k(X, \Lambda) = \min_{\phi \in C^k(X, \Lambda) \setminus Z^k(X, \Lambda)} \frac{\|d_k\phi\|}{\min_{\psi \in Z^k(X, \Lambda)} \text{dist}(\phi, \psi)}.$$

Observe the following (for $k = 0, 1$):

- For any X, $h_{\mathrm{cs}}^k(X, \Lambda) > 0$.

- If $H^k(X, \Lambda)$ is non-trivial, then $h_{\mathrm{cb}}^k(X, \Lambda) = 0$.

- If $H^k(X, \Lambda)$ is trivial, then $h_{\mathrm{cb}}^k(X, \Lambda) = h_{\mathrm{cs}}^k(X, \Lambda)$.

For a constant $\beta > 0$, we say that

- The complex X is a (Λ, β) 1-*coboundary expander*, if $h_{\mathrm{cb}}^0(X, \Lambda) \geq \beta$ and $h_{\mathrm{cb}}^1(X, \Lambda) \geq \beta$.

- The complex X is a (Λ, β) 1-*cosystolic expander*, if $h_{\mathrm{cs}}^0(X, \Lambda) \geq \beta$, $h_{\mathrm{cs}}^1(X, \Lambda) \geq \beta$ and for every $\phi \in Z^1(X, \Lambda) \setminus B^1(X, \Lambda)$ it holds that $\|\phi\| \geq \beta$.

We say that a family of pure n-dimensional simplicial complexes $\{X^{(s)}\}_s$, is a family of 1-*coboundary expanders with Λ coefficients* (1-*cosystolic expanders with Λ coefficients*), if the the following holds:

- The family $\{X^{(s)}\}_s$ is of bounded degree, i.e.,

$$\sup_s \sup_{v \in X^{(s)}(0)} |\{\sigma \in X(n) : v \in \sigma\}| < \infty.$$

- The size of $X^{(s)}$ grows with s, i.e.,

$$\lim_s |X^{(s)}(0)| = \infty.$$

- There exists $\beta > 0$ such that for every s, $X^{(s)}$ is a (Λ, β) 1-coboundary expander ((Λ, β) 1-cosystolic expander).

Remark 2.1. A lower bound on h_{cb}^0 is usually easy to verify: If the 1-skeleton of X is a connected finite graph, then for any non-trivial group Λ, $h_{\mathrm{cb}}^0(X, \Lambda)$ is exactly the Cheeger constant of the weighted 1-skeleton (where each edge $\{u, v\}$ is weighted by $w(\{u, v\})$). Moreover, the Cheeger constant of the weighted 1-skeleton can be bounded by the spectral gap of the weighted random walk on the 1-skeleton, which in concrete examples can be bounded using the trickling down Theorem from [Opp18].

In [DD24a], the following result was proven (generalizing previous results in [KKL14, EK24] that applied to the Abelian setting, and improving parameters over the result of [KM21], that was the first cosystolic expansion result in the non-Abelian setting):

Theorem 2.2. *[DD24a, Theorem 8]* Let $0 \leq \lambda < 1, \beta > 0$ be constants and Λ be a group. For a finite pure n-dimensional simplicial complex X with $n \geq 3$, if for every vertex v, $h_{\mathrm{cb}}^1(\mathrm{lk}_X(v), \Lambda) \geq \beta$ and X is a λ-one-sided local spectral expander, then

$$h_{\mathrm{cs}}^1(X, \Lambda) \geq \frac{(1 - \lambda)\beta}{24} - e\lambda,$$

where $e \approx 2.71$ is the Euler constant.

2.2 Kac–Moody–Steinberg groups and complexes

We recall the notions of generalized Cartan matrices and root systems which are the building blocks of Chevalley, Steinberg and Kac-Moody-Steinberg groups. We then describe the construction of KMS complexes and explain how we get KMS complexes over Chevalley groups, called congruence KMS complexes, which will be the objects of our main result, Corollary 5.3.

2.2.1 Root systems and Chevalley groups

We start by recalling some facts about generalized Cartan matrices, the associated root systems and Chevalley groups. These are the building blocks of KMS groups.

Most of the structure of the KMS groups is encoded in its generalized Cartan matrix, which also gives rise to a Dynkin diagram and a root system.

Definition 2.3. A generalized Cartan matrix (GCM) is a matrix $A = (A_{ij})_{i,j \in I} \in \mathrm{Mat}_n(\mathbb{Z})$ such that $A_{ii} = 2$ for all $i \in I$, $A_{ij} \leq 0$ for all $i \neq j \in I$ and $A_{ij} = 0 \iff A_{ji} = 0$. We call $|I|$ the rank of A.

Every GCM $A = (A_{ij})_{i,j \in I}$ gives rise to a Dynkin diagram in the following way. As vertex set, we take the index set I and two vertices i, j are connected by $|A_{ij}|$ edges if $A_{ij}A_{ji} \leq 4$ and $|A_{ij}| \geq |A_{ji}|$, and these edges are equipped with an arrow pointing towards i if $|A_{ij}| > 1$. If $A_{ij}A_{ji} > 4$, the vertices i and j are connected by a bold-faced labelled edge with the ordered pair of integers $|A_{ij}|, |A_{ji}|$.

A GCM A is *irreducible* if there exists no non-trivial partition $I = I_1 \cup I_2$ such that $A_{i_1 i_2} = 0$ for all $i_1 \in I_1, i_2 \in I_2$.

Definition 2.4. Let A be a $(d+1) \times (d+1)$ generalized Cartan matrix with index set $I = \{0, \ldots, d\}$, and let $J \subseteq I$.

(i) The subset J is called *spherical* if $A_J = (A_{ij})_{i,j \in J}$ is of spherical type, meaning that the associated Coxeter group (see e.g. [Mar18, Proposition 4.22]) is finite (see e.g. [Bou08, Chapter 6.4.1]). Given $n \geq 2$, A is *n-spherical* if every subset $J \subseteq I$ of size n is spherical.

(ii) We denote by Q_A the set of spherical subsets of I associated to the generalized Cartan matrix A.

(iii) A generalized Cartan matrix A is *purely n-spherical* if every spherical subset $J \subseteq I$ is contained in a spherical subset of size n. (In particular, no set of size $n+1$ is spherical for a purely n-spherical generalized Cartan matrix.)

(iv) Analogously, we call a GCM *n-classical* if each irreducible factor of $A_J = (A_{ij})_{i,j \in J}$ is of classical type A_k, B_k, C_k, D_k (see e.g. [Bou08, Chapter 6.4.1]) for each $J \subseteq I$ with $|J| \leq n$.

(v) To a generalized Cartan matrix A we can associate the following sets:

- a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$,
- a set of real roots $\Phi \subseteq \bigoplus_{i \in I} \mathbb{Z}\alpha_i$,
- two sets, one of positive and one of negative real roots: $\Phi^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i \cap \Phi, \Phi^- = -\Phi^+$.

Note that $\Phi = \Phi^+ \sqcup \Phi^-$. More details can be found e.g. in [Mar18, Chapter 3.5].
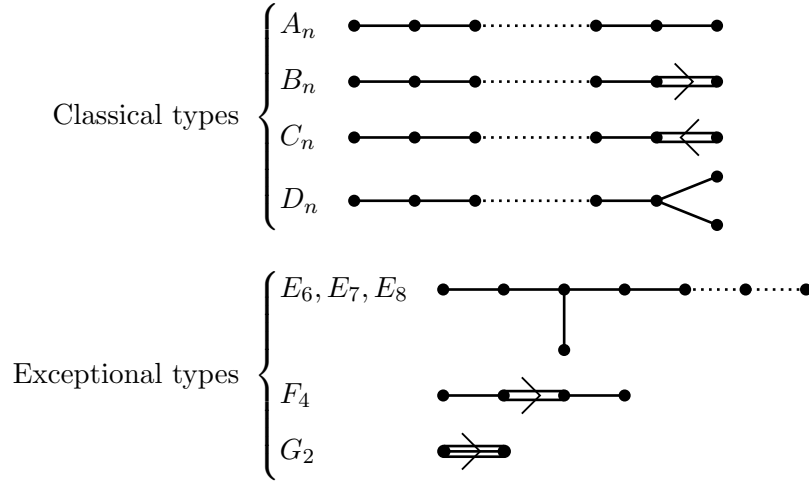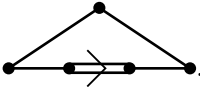
Figure 1: The Dynkin diagrams of irreducible spherical root systems

Remark 2.5. (i) Irreducible spherical root systems have been classified. All possible diagrams are given in Figure 1.

(ii) All rank $n+1$, n-spherical diagrams have been classified and are precisely the affine diagrams and the compact hyperbolic diagrams, the latter have rank at most 5, where the diagram of rank 5 is exactly the one that we have to exclude to get n-classicality. See e.g. [Hum90, Chapter 6.9] for the classification result.

The non-spherical, rank $n+1$, n-classical diagrams, with $n \geq 3$ are precisely the non-spherical rank $n+1$ diagrams which are n-spherical but not one of the following

- affine diagrams $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$,

- compact hyperbolic diagram .

Definition 2.6. Corresponding to any irreducible, spherical GCM $\mathring{A}$ with root system $\mathring{\Phi}$ of rank at least 2, and any finite field K, there is an associated universal (or simply connected) Chevalley group, denoted $\mathrm{Chev}_{\mathring{A}}(K)$. Abstractly, it is generated by symbols $x_\alpha(s)$ for $\alpha \in \mathring{\Phi}$ and $s \in K$, subject to the relations

$$x_\alpha(s)x_\alpha(u) = x_\alpha(s+u)$$

$$[x_\alpha(s), x_\beta(u)] = \prod_{i,j>0} x_{i\alpha+j\beta}\left(C_{ij}^{\alpha,\beta}s^i u^j\right) \quad \text{(for } \alpha+\beta \neq 0\text{)}$$

$$h_\alpha(s)h_\alpha(u) = h_\alpha(su) \quad (\text{ for } s,u \neq 0),$$

$$\text{where} \quad h_\alpha(s) = n_\alpha(s)n_\alpha(-1)$$

$$\text{and} \quad n_\alpha(s) = x_\alpha(s)x_{-\alpha}\left(-s^{-1}\right)x_\alpha(s).$$

Note that the $C_{ij}^{\alpha,\beta}$ are integers called structure constants that can be found in [Car89].

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Remark 2.7. (i) Let K be a field and let $K[t]$ denote the polynomial ring in one variable over K. The simply-connected Chevalley group $\mathrm{Chev}_{\mathring{A}}(K[t])$ is, similar to the case of a Chevalley group over a field, generated by elements $x_\alpha(s)$ for $\alpha \in \mathring{\Phi}, s \in K[t]$ that satisfy the relations above, where for the third relation we have to add the extra assumption that u, s are invertible in $K[t]$. This can be found e.g. in [Reh75].

(ii) More generally, one can define Chevalley groups by introducing the *Chevalley–Demazure group scheme*, which is a functor from the category of commutative unital rings to the category of groups. Over fields and their polynomial ring in one variable these definitions yield the same groups.

Chevalley groups are closely related to Steinberg groups, which are defined as follows.

Definition 2.8. Given an irreducible, spherical Cartan matrix $\mathring{A}$ with root system $\mathring{\Phi}$ of rank at least 2, and a ring R, we define the Steinberg group of type $\mathring{A}$ over R to be the group given by the following presentation.

$$\mathrm{St}(R) := \mathrm{St}(\mathring{A}, R) := \langle x_\alpha(u); \alpha \in \mathring{\Phi}, u \in R \mid \mathcal{R} \rangle$$

where the relations $\mathcal{R}$ are

$$x_\alpha(s)x_\alpha(u) = x_\alpha(s+u) \quad \text{for } \alpha \in \mathring{\Phi}, u, s \in R,$$
$$[x_\alpha(s), x_\beta(u)] = \prod_{i,j>0} x_{i\alpha+j\beta}\left(C_{ij}^{\alpha,\beta} s^i u^j\right) \quad \text{for } \alpha, \beta \in \mathring{\Phi}, \alpha + \beta \neq 0, s, u \in R^\times.$$

Remark 2.9. Note that all the relations of the Steinberg group also hold in the corresponding Chevalley group, while the Chevalley group has potentially more relations.

Over finite fields the Chevalley group and Steinberg group coincide, see e.g. [Wei13, Corollary 6.1.1].

2.2.2 Kac–Moody–Steinberg groups

We can now put the ingredients together to define what a Kac–Moody–Steinberg group is and present some of its properties. For more details, see [GdPVB25, Chapter 3] and references therein.

Definition 2.10. Let K be a field. Let $A = (A_{ij})_{i,j\in I}$ be a GCM over the index set I which is 2-spherical. Let $Q_A = \{J \subseteq I \mid A_J := (A_{ij})_{i,j\in J}$ is spherical$\}$. Let Φ be the (real) root system associated to A with simple roots $\Pi = \{\alpha_i \mid i \in I\}$. For $J \in Q_A$ set

$$U_J := \langle x_{\alpha_i}(s) \mid s \in K, i \in J \rangle \leq \mathrm{Chev}_{A_J}(K)$$

the group generated by all simple roots in the Chevalley group of type A_J over K. Note that if $L \subset J$ then we have a natural inclusion $U_L \hookrightarrow U_J$ by sending $x_{\alpha_i}(s) \in U_L$ to the same generator in U_J. The KMS group of type A over K is defined as the free product of the $U_J, J \in Q_A$ modulo the natural inclusions:

$$\mathcal{U}_A(K) = \mathop{\Large*}_{J \in Q_A} U_J / (U_L \hookrightarrow U_J, L \subseteq J).$$

Note that $U_J \hookrightarrow \mathcal{U}_A(K)$ and we will denote the image of U_J in $\mathcal{U}_A(K)$ again by U_J. These subgroups are called the local groups of $\mathcal{U}_A(K)$. For $i, j \in I$ we will write $U_i := U_{\{i\}}$ and $U_{i,j} = U_{\{i,j\}}$.

Remark 2.11. For two roots $\alpha, \beta \in \Phi$ we write $]\alpha, \beta[_{\mathbb{N}} = \{n_1\alpha + n_2\beta \in \Phi \mid n_1, n_2 \in \mathbb{N}^*\}$ and $[\alpha, \beta]_{\mathbb{N}} =]\alpha, \beta[_{\mathbb{N}} \cup \{\alpha, \beta\}$.

We have the following abstract presentation for the KMS groups:

$$\mathcal{U}_A(K) = \left\langle u_\beta(t) \text{ for } t \in K, i, j \in I, \beta \in [\alpha_i, \alpha_j]_{\mathbb{N}} \mid \mathcal{R} \right\rangle$$

where the set of relations $\mathcal{R}$ is defined as:

for all $i, j \in I$, $\{\alpha, \beta\} \subseteq [\alpha_i, \alpha_j]_{\mathbb{N}}$, $t, u \in K$:

$$u_\alpha(t)u_\alpha(s) = u_\alpha(t + s)$$

$$[u_\alpha(t), u_\beta(u)] = \prod_{\gamma = k\alpha + l\beta \in]\alpha, \beta[_{\mathbb{N}}} u_\gamma\left(C_{k,l}^{\alpha\beta} t^k u^l\right).$$

The constants $C_{k,l}^{\alpha,\beta}$ are the same structure constants as in the presentation of the Chevalley group.

Since A is 2-spherical, the subgroups U_β, for $\beta \in [\alpha_i, \alpha_j]$, are contained in the group generated by the root groups $U_{\alpha_i}, U_{\alpha_j}$ (see [Abr96, Proposition 7]).

2.2.3 KMS complexes

We now use the KMS groups defined above in the coset complex construction to get pure simplicial complexes called KMS complexes, which were first introduced in [GdPVB25]. First, we recall the definition of coset complexes and state one key fact. More details on coset complexes can be found for example in [KO23].

Definition 2.12. Let G be a group and $\mathcal{H} = (H_0, \ldots, H_n)$ be a family of subgroups of G. Then the coset complex $\mathcal{CC}(G; (H_0, \ldots, H_n))$ is defined to be the simplicial complex with

- vertex set $\bigsqcup_{i=0}^{n} G/H_i$

- a set of vertices $\{g_1 H_{i_1}, \ldots, g_k H_{i_k}\}$ forms a $(k-1)$-simplex in $\mathcal{CC}(G; \mathcal{H})$ if and only if $\bigcap_{j=1}^{k} g_j H_{i_j} \neq \emptyset$.

Note that the set of vertices is partitioned into $n+1$ subsets G/H_i for $i = 0, \ldots, n$ such that for any simplex $\sigma \in \mathcal{CC}(G; \mathcal{H})$ we have $|\sigma \cap G/H_i| \leq 1$ for all i. In that case, we say that $\mathcal{CC}(G; \mathcal{H})$ is a $(n+1)$-partite simplicial complex.

Definition 2.13. The *type* of a simplex $\{g_1 H_{i_1}, \ldots, g_k H_{i_k}\}$ in $\mathcal{CC}(G; \mathcal{H})$ is the set of indices $\{i_1, \ldots, i_k\} \subseteq \{0, \ldots, n\}$. Given a type $\emptyset \neq T \subseteq \{0, \ldots, n\}$ we write

$$H_T := \bigcap_{i \in T} H_i \qquad \text{and set} \qquad H_\emptyset := \langle H_0, \ldots, H_n \rangle \leq G.$$

The following well-known fact characterizes links of faces in a coset complex.

Fact 2.14. Let σ be a face in $\mathcal{CC}(G, \mathcal{H})$ of type $T \neq \emptyset$. Then the link of σ is isomorphic to the coset complex $\mathcal{CC}\left(H_T, \left(H_{T \cup \{i\}} : i \notin T\right)\right)$.

KMS complexes are coset complexes over certain finite quotients of KMS groups. The precise definition is as follows:

Definition 2.15. Let A be a GCM over the index set I which is $(|I|-1)$-spherical but non-spherical. Let K be a finite field of size $q \geq 4$. Let $\phi : \mathcal{U}_A(K) \to G$ be a finite quotient of $\mathcal{U}_A(K)$ such that

(i) $\phi|_{U_J}$ is injective for all $J \in Q_A$ (i.e. ϕ is injective on the local groups),

(ii) $\phi(U_J \cap U_L) = \phi(U_J) \cap \phi(U_L)$ for all $J, L \in Q_A$.

Then we set

$$X = \mathcal{CC}(G; (\phi(U_{I \setminus \{j\}}))_{j \in I})$$

and call X a KMS complex of type A over K.

It is know that KMS complexes satisfying some mild assumptions are local-spectral expander:

Theorem 2.16. *[GdPVB25, Theorem 4.3] Let A be a GCM over the index set I which is non-spherical but $(|I| - 1)$-spherical. Let K be a finite field of size $q \geq 4$. Let $\phi_i : \mathcal{U}_A(K) \to G_i, i \in I$ be family of finite quotients of $\mathcal{U}_A(K)$ such that $X_i = \mathcal{CC}(G_i, (\phi_i(U_{I \setminus \{j\}}))_{j \in I}), i \in \mathbb{N}$ is a family of KMS complexes and $|G_i| \overset{i \to \infty}{\longrightarrow} \infty$. If q is such that $\sqrt{\frac{3}{q}} \leq \frac{1}{|I|-1}$ then the family $(X_i)_{i \in \mathbb{N}}$ is a family of bounded degree λ-local spectral expanders for some $\lambda = \lambda(q) < 1$.*

Moreover, if $q \gg (|I|-1)^2$, then the family $(X_i)_{i \in \mathbb{N}}$ is a family of bounded degree $\frac{2}{\sqrt{q}}$-local spectral expanders.

It was proven in [OVB25] that the $(n-1)$-skeleton of an n-classical KMS complex (over a large enough field) is a cosystolic expander with Abelian coefficients.

2.2.4 Congruence KMS complexes

In this part, we explore the relation between affine KMS groups and the corresponding Chevalley groups, following the results of [GdPVB25, Chapter 6]. We will repeat an explicit construction of a family of simplicial complexes given in [GdPVB25, Chapter 6] which we will call here *congruence KMS complexes* (this terminology did not appear in [GdPVB25]).

Let $\mathring{A} = (A_{ij})_{i,j \in \mathring{I}}, \mathring{I} = \{1, \ldots n\}$ be an irreducible spherical GCM with spherical root system $\mathring{\Phi}$ and set of simple roots $\mathring{\Pi} = \{\alpha_1, \ldots, \alpha_n\}$. This root system has a unique highest root $\gamma \in \mathring{\Phi}$, which means that γ is such that for any $\alpha_i \in \mathring{\Pi}$ we have $\gamma + \alpha_i \notin \mathring{\Phi}$.

Note that, whenever we remove one root from the set $S := \{-\gamma, \alpha_1, \ldots, \alpha_n\}$, we get a set of simple roots (although they might not generate the full root system $\mathring{\Phi}$ but a subsystem).

We set $\alpha_0 = -\gamma$ and $I = \{0, \ldots, n\}$. The idea now is to consider the set $\Pi = \{\overline{\alpha}_0, \ldots, \overline{\alpha}_n\}$ as set of simple roots of a root system Φ such that for all $i, j \in I$ the roots $\overline{\alpha}_i, \overline{\alpha}_j$ generate the same rank two subsystem as α_i, α_j in $\mathring{\Phi}$. The resulting root systems Φ with simple roots $\Pi = \{\overline{\alpha}_0, \ldots, \overline{\alpha}_n\}$ and generalized Cartan matrix A are described by the Dynkin diagrams in [Kac83, §4.8] (or [Mar18, Table 5.1 Aff 1]) and are of affine (untwisted) type. Note that A is of rank $n + 1$, n-spherical and purely n-spherical. If $\mathring{A}$ is of classical type, then A is n-classical.

For a root $\beta = \sum_{i \in \mathring{I}} \lambda_i \alpha_i \in \mathring{\Phi}$, $\lambda_i \in \mathbb{Z}$, we write $\overline{\beta} = \sum_{i \in \mathring{I}} \lambda_i \overline{\alpha}_i \in \Phi$. We set $\overline{\delta} = \overline{\alpha}_0 + \overline{\gamma} \in \bigoplus_{i \in I} \mathbb{Z}\overline{\alpha}_i$ which is not in Φ but it is what is called an imaginary root of the root system associated to A. For our purposes, it suffices to think of it as an element in $\bigoplus_{i \in I} \mathbb{Z}\overline{\alpha}_i$. From [Kac83, Section 7.4] it follows that $\Phi = \{a_{\alpha,m} := \overline{\alpha} + m\overline{\delta} \mid m \in \mathbb{Z}, \alpha \in \mathring{\Phi}\}$. Note that $\overline{\alpha}_0 = a_{-\gamma,1}$. A root $a_{\alpha,m}$ is in Φ^+ if and only if $m \geq 1$ or $m = 0$ and $\alpha \in \mathring{\Phi}^+$ (see e.g. [Mar18, proof of Theorem 7.90]).

Before we continue with the construction, we fix a few notations: $K[t]$ denotes the polynomial ring in one variable t over a field K. For a polynomial $f \in K[t]$, let $(f) = \{g \cdot f \mid g \in K[t]\}$ denote the ideal generated by f in $K[t]$ (e.g. (t) is the set of polynomials without constant term) and let $Kt^n = \{\lambda t^n \mid \lambda \in K\}$ be the set of all scalar multiples of t^n.

Let K be a finite field of order $|K| \geq 5$ with $\mathrm{char}(K) \neq 2$ and assume that $\mathrm{rank}(\mathring{\Phi}) \geq 3$ (thus we have that $\mathcal{U}_A(K)$ is isomorphic to the maximal positive unipotent subgroup of the corresponding Kac–Moody group, see [GdPVB25, Proposition 3.8]). The aforementioned result, together with [Mar18, Theorem 7.90] imply that the following is a well-defined injective homomorphism

$$\phi : \mathcal{U}_A(K) \to \mathrm{Chev}_{\mathring{A}}(K[t])$$
$$u_{a_{\alpha,m}}(\lambda) \mapsto x_\alpha(\lambda t^m).$$

Let $f \in K[t]$ be an irreducible polynomial of degree $\ell \geq 2$. We denote by $\pi_f : K[t] \to K[t]/(f)$ the projection to the quotient of the polynomial ring by the ideal generated by f.

Using the functoriality of the Chevalley group functor, we can go from the Chevalley group over $K[t]$ to the one over $K[t]/(f)$ by applying π_f "entry-wise". We get the following well-defined map

$$\varphi_f : \mathcal{U}_A(K) \xrightarrow{\phi} \mathrm{Chev}_{\mathring{A}}(K[t]) \xrightarrow{\pi_f} \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$$
$$u_{\alpha_i}(\lambda) \mapsto \begin{cases} x_{\alpha_i}(\lambda) & i \neq 0 \\ x_{-\gamma}(\lambda t) & i = 0 \end{cases} \mapsto \begin{cases} x_{\alpha_i}(\lambda + (f)) & i \neq 0 \\ x_{-\gamma}(\lambda t + (f)) & i = 0 \end{cases}.$$

In [GdPVB25, Chapter 6] it was shown that the maps φ_f satisfy the assumptions of Definition 2.15, hence give rise to KMS complexes. In particular, choosing an infinite family of irreducible polynomials with increasing degree gives rise to a family of bounded degree high-dimensional spectral expanders, see [GdPVB25, Theorem 5.13].

This leads to the following definition of congruence KMS complexes:

Definition 2.17 (Congruence KMS complexes). For every spherical root system $\mathring{\Phi}$ of type $\mathring{A}$ and rank n, every finite field K of order ≥ 5 and characteristic different from 2, and every irreducible polynomial $f \in K[t]$ of degree ≥ 2, we define the congruence KMS complex $X(\mathring{\Phi}, K, f)$ as

$$X(\mathring{\Phi}, K, f) = \mathcal{CC}(\mathrm{Chev}_{\mathring{A}}(K[t]/(f)); (\pi_f(H_i))_{i \in \{0,\ldots,n\}})$$

where $\pi_f : \mathrm{Chev}_{\mathring{A}}(K[t]) \to \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$ is the projection induced by the map $K[t] \to K[t]/(f)$, and for $i = 1, \ldots, n$ we set $H_i = \langle x_{-\gamma}(\lambda t), x_{\alpha_j}(\lambda) \mid j \in \{1, \ldots, n\} \setminus \{i\}, \lambda \in K \rangle \leq \mathrm{Chev}_{\mathring{A}}(K[t])$ and $H_0 = \langle x_{\alpha_j}(\lambda) \mid j \in \{1, \ldots, n\}, \lambda \in K \rangle \leq \mathrm{Chev}_{\mathring{A}}(K[t])$.

Given a sequence of irreducible polynomials $f_s \in K[t]$ of degree $\ell \geq 2$ such that $\deg(f_s)$ tends to infinity, we define the family of congruence KMS complexes associated to the sequence $\{f_s\}$ to be the family $\{X^{(s)} = X(\mathring{\Phi}, K, f_s)\}_s$.

Last, we define congruence KMS complexes of type X over a field K to be any family of congruence KMS complexes $\{X^{(s)} = X(\mathring{\Phi}, K, f_s)\}_s$ associated to some sequence $\{f_s\}$ as above where $\mathring{\Phi}$ is of type X (e.g., $\mathring{\Phi}$ is of type $X = A_n$).

3 Vanishing of first cohomology

In this chapter, we show that for the congruence KMS complexes the first cohomology with many coefficient groups vanishes. Together with the results from Section 5 this shows that the congruence KMS complexes give rise to families of bounded degree 1-coboundary expanders.

We want to apply [KOW24, Theorem 3.1] to our setting. We recall the statement:

Theorem 3.1 ([KOW24])**.** *Let X be a connected pure 2-dimensional simplicial complex and N be a group acting on X. Assume $p : X \to N \backslash X$ is rigid, i.e. it is simplicial and for all $\sigma \in X$ we have $|\sigma| = |p(\sigma)|$. For every group Λ, if $H^1(X, \Lambda) = 0$ and $H^1(N, \Lambda) = 0$ then $H^1(N \backslash X, \Lambda) = 0$ (the cohomology of N is taken with the trivial action of N on Λ).*

In the remainder of the section, we will first investigate the underlying infinite complex in our setting, and then the kernel of the quotient map.

3.1 The underlying infinite complex

The complex that will play the role of X in Theorem 3.1 is the following.

Definition 3.2. Given a n-spherical, purely n-spherical GCM $A = (A_{ij})_{i,j \in I}$ of rank $n + 1$ and a field K, we define the *fundamental KMS complex* to be

$$Y_A(K) = \mathcal{CC}(\mathcal{U}_A(K), (U_{I \backslash \{i\}})_{i \in I}).$$

Observation 3.3. The complex $Y_A(K)$ is a connected, pure, n-dimensional simplicial complex.

Proposition 3.4. *Let A be a n-spherical, purely n-spherical GCM of rank $n + 1$, then $Y_A(K)$ is simply connected, in particular $H^1(Y_A(K), \Lambda) = 0$ for every group Λ.*

Proof. This follows from [AH93, Theorem 2.4] since $\mathcal{U}_A(K)$ can be described as the free product of the $U_{I \backslash \{i\}}$ amalgamated along their intersections. $\square$

Lemma 3.5. *Given $\varphi_f : \mathcal{U}_A(K) \to \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$ as in Section 2.2.4. Let $N_f = \ker \varphi_f$. Then $N_f \backslash Y_A(K) \cong \mathcal{CC}(\mathrm{Chev}_{\mathring{A}}(K[t]/(f)), (\varphi_f(U_{I \backslash \{i\}})_{i \in I})$ and $p_f : Y_A(k) \to N_f \backslash Y_A(K)$ is rigid.*

Proof. This follows from [KOW24, Proposition 2.5] and [KOW24, Observation 2.2]. $\square$

3.2 The kernel of the quotient map

Let A by a generalized Cartan matrix of affine type and rank $n \geq 4$, and let K be a finite field with $|K| \geq 5$. For simplicity, we assume that $\mathrm{char}(K) \neq 2$, see [GdPVB25, Remark 5.10]. Let $\mathring{A}$ denote the spherical GCM related to A as in the construction in Section 2.2.4, with root system $\mathring{\Phi}$. Recall from Section 2.2.4 that we have an injective homomorphism

$$\phi : \mathcal{U}_A(K) \to \mathrm{Chev}_{\mathring{A}}(K[t])$$

$$u_{\alpha_i}(\lambda) \mapsto \begin{cases} x_{\alpha_i}(\lambda) & i \neq 0 \\ x_{-\gamma}(\lambda t) & i = 0 \end{cases}$$

where γ is the longest root in $\mathring{\Phi}$.

Let $\mathcal{I}$ denote the image of ϕ in $\mathrm{Chev}(K[t])$ (since the type of the Chevalley group is fixed here, we will omit if from the notation). As shown in [GdPVB25, Proposition 5.11], we have

$$\mathcal{I} = \langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in K[t], \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (t) \rangle \leq \mathrm{Chev}(K[t]).$$

Furthermore, let $f \in K[t]$ be irreduible with $\deg(f) \geq 2$.

The quotient map $K[t] \to K[t]/(f)$ induces the map $\pi_f : \mathrm{Chev}(K[t]) \to \mathrm{Chev}(K[t]/(f))$. We will denote its kernel by $\mathrm{Chev}(K[t], (f)) := \ker(\pi_f)$.

We introduce some further notation, where for a group G and a subset S we denote by $\langle\langle S \rangle\rangle_G$ the normal subgroup of G generated by S.

$$\mathcal{I}(f) = \langle\langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in (f), \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (tf) \rangle\rangle_{\mathcal{I}},$$

$$\mathrm{St}\,(\mathcal{I}) = \langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in K[t], \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (t) \rangle \leq \mathrm{St}\,(K[t]),$$

$$\mathrm{St}\,(\mathcal{I}, (f)) = \langle\langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in (f), \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (tf) \rangle\rangle_{\mathrm{St}(\mathcal{I})}.$$

The following result is of a similar flavor as [ACR12, Lemma 3.3] and the proof is inspired by the proof of the aforementioned result.

Theorem 3.6. *In the above set up, we have*

$$\mathrm{Chev}(K[t], (f)) \cap \mathcal{I} = \mathcal{I}(f).$$

Proof. We have the following commutative diagram.

$$
\begin{array}{ccc}
0 & \xrightarrow{\ \ \alpha\ \ } & 0 \\
\downarrow & & \downarrow \\
\mathrm{St}(\mathcal{I}, (f)) & \xrightarrow{\ \ \beta\ \ } & \mathrm{Chev}(K[t], (f)) \cap \mathcal{I} \\
\downarrow & & \downarrow \\
\mathrm{St}(\mathcal{I}) & \xrightarrow{\ \ \gamma\ \ } & \mathcal{I} \\
\downarrow & & \downarrow \\
\mathrm{St}(K[t]/(f)) & \xrightarrow{\ \ \delta\ \ } & \mathrm{Chev}(K[t]/(f))
\end{array}
$$

Here, α is the identity, β, γ and δ are given by mapping a generator $x_\alpha(g)$ of the Steinberg group to the corresponding generator (with the same label) in the Chevalley group.

Note that $\mathcal{I}(f) = \beta(\mathrm{St}(\mathcal{I}, (f))) \subseteq \mathrm{Chev}(K[t], (f)) \cap \mathcal{I}$.

The vertical maps in the diagram are first the trivial map, then the inclusion map followed by the quotient maps $\mathrm{St}(K[t]) \to \mathrm{St}(K[t]/(f))$ and $\mathrm{Chev}(K[t]) \to \mathrm{Chev}(K[t]/(f))$ respectively, both restricted to the desired domain.

Our goal is to show that β is surjective. To this end, we want to apply the "Four lemma" (see e.g. [ML63, Lemma I.3.2]) which says that if the columns are exact, δ is injective and α and γ are surjective then β is surjective.

Note that α is trivially surjective, δ is an isomorphism, since $K[t]/(f)$ is a finite field (see Remark 2.9) and γ is surjective since all generators of $\mathcal{I}$ lie in the image of $\mathrm{St}(\mathcal{I})$.

It remains to show the exactness of the columns. For the right column, this follows from the definitions.

For the left column we need to show that $M := \ker(\mathrm{St}\,(\mathcal{I}) \to \mathrm{St}\,(K[t]/(f)))$ is equal to $N := \mathrm{St}\,(\mathcal{I}, (f))$.

Clearly we have that $N \subseteq M$. For the other inclusion, we proceed in two steps.

Step 1: We show that $\mathrm{St}\,(\mathcal{I})\,/N \cong \mathrm{St}\,(\mathcal{I})\,/M$.

Thus, let $\psi : \mathrm{St}\,(\mathcal{I}) \to \mathrm{St}\,(\mathcal{I})\,/N$ denote the projection. For $\alpha \in \mathring{\Phi}^+, g \in K[t], h \in (f)$ we have

$$\psi(x_\alpha(g+h)) = \psi(x_a(g))\psi(x_\alpha(h)) = \psi(x_\alpha(g))$$

since $x_\alpha(h) \in N$. Similarly, for $\alpha \in \mathring{\Phi}^-, g \in (t), h \in (tf)$ we have

$$\psi(x_\alpha(g+h)) = \psi(x_a(g))\psi(x_\alpha(h)) = \psi(x_\alpha(g)).$$

Thus the expressions $\psi(x_\alpha(g+(f)))$ for $\alpha \in \mathring{\Phi}^+, g \in K[t]$ and $\psi(x_\alpha(g+(tf)))$ for $\alpha \in \mathring{\Phi}^-, g \in (t)$ are well defined. In particular, they form a set of generators of $\mathrm{St}\,(\mathcal{I})\,/N$.

Let $\mathrm{ht}(\alpha)$ denote the height of the root α with respect to a fixed set of simple roots (i.e. if $\mathring{\Pi} = \{\alpha_1, \dots, \alpha_n\}$ and $\alpha = \sum_{i=1}^n \lambda_i \alpha_i$ then $\mathrm{ht}(\alpha) = \sum_{i=1}^n \lambda_i$). Note that if $\alpha \in \mathring{\Phi}^+$ then $\mathrm{ht}(\alpha) > 0$ and if $\alpha \in \mathring{\Phi}^-$ then $\mathrm{ht}(\alpha) < 0$ (and the height is always an integer).

We define the following map:

$$\xi : \mathrm{St}\,(K[t]/(f)) \to \mathrm{St}\,(\mathcal{I})\,/N$$

$$x_\alpha(g+(f)) \mapsto \begin{cases} \psi(x_\alpha(t^{-\mathrm{ht}(\alpha)}g + (f))) & \text{if } \alpha \in \mathring{\Phi}^+ \\ \psi(x_\alpha(t^{-\mathrm{ht}(\alpha)}g + (tf))) & \text{if } \alpha \in \mathring{\Phi}^- \end{cases}.$$

Note that since $K[t]/(f)$ is a field, it is well defined to consider t^{-1} modulo (f). We need to check that ξ is a well-defined homomorphism. Let $g \in (f)$. Then $t^{-\mathrm{ht}(\alpha)}g \in (f)$ if α is positive and $t^{-\mathrm{ht}(\alpha)}g \in (tf)$ if α is negative. Hence different representatives of the same class $g + (f)$ get mapped to the same element. Next, we check that the relations are respected by the map.

Clearly, $\xi(x_\alpha(s+u)) = \xi(x_\alpha(s))\xi(x_\alpha(u))$.

Note that $\mathrm{ht}(i\alpha + j\beta) = i\,\mathrm{ht}(\alpha) + j\,\mathrm{ht}(b)$ for all $\alpha, \beta \in \mathring{\Phi}, i, j \in \mathbb{Z}$ and hence

$$[x_\alpha(t^{-\mathrm{ht}(\alpha)}s), x_\beta(t^{-\mathrm{ht}(\beta)}u)] = \prod_{i\alpha+j\beta \in \mathring{\Phi}, i,j>0} x_{i\alpha+j\beta}(C_{i,j}^{\alpha,\beta} t^{-\mathrm{ht}(i\alpha+j\beta)} s^i u^j).$$

This implies that the commutator relation is also preserved under ξ.

Hence, ξ is a well-defined homomorphism. The next step is to show that it is surjective. This follows from the fact that

$$K[t]/(f) \to K[t]/(f) : \lambda \mapsto (t+(f))^n \lambda$$

is bijective for all $n \in \mathbb{Z}$ and that the map

$$K[t]/(f) \to (t)/(tf) : \lambda \mapsto t\lambda$$

is well defined and bijective. Hence all the generators of $\mathrm{St}\,(\mathcal{I})\,/N$ lie in the image of ξ.

On the other hand, since $N \subseteq M$, the surjective map $\mathrm{St}\,(\mathcal{I}) \to \mathrm{St}\,(K[t]/(f))$ induces a surjective map $\mathrm{St}\,(\mathcal{I})\,/N \to \mathrm{St}\,(K[t]/(f))$. Since $\mathrm{St}\,(K[t]/(f))$ is finite and $\mathrm{St}\,(K[t]/(f)) \cong \mathrm{St}\,(\mathcal{I})\,/M$ we get that $\mathrm{St}\,(\mathcal{I})\,/N \cong \mathrm{St}\,(\mathcal{I})\,/M$, which concludes Step 1.

Step 2: Since $\mathrm{St}\,(\mathcal{I})\,/M \cong \mathrm{St}\,(K[t]/(f)) \cong \mathrm{Chev}(K[t]/(f))$ is finite, we can conclude from Step 1 that $N = M$.

To show this, note that we have $N, M \lhd \mathrm{St}\,(\mathcal{I}), N \subseteq M$ thus $M/N \lhd \mathrm{St}\,(\mathcal{I})\,/N$. Hence by the third isomorphism theorem $(\mathrm{St}\,(\mathcal{I})\,/N)/(M/N) \cong \mathrm{St}\,(\mathcal{I})\,/M$. Hence $|\mathrm{St}\,(\mathcal{I})\,/N|/|M/N| = |\mathrm{St}\,(\mathcal{I})\,/M|$. But $|\mathrm{St}\,(\mathcal{I})\,/N| = |\mathrm{St}\,(\mathcal{I})\,/M|$, thus $|M/N| = 1$ which implies $N = M$.

$\square$

Corollary 3.7. *Assume the setting from above. Let $p = \mathrm{char}(K)$ and let Λ be a group with no non-trivial elements of order p. Then $H^1(\ker(\varphi_f), \Lambda) = 0$, where the cohomology is considered with trivial action of $\ker(\varphi_f)$ on Λ.*

Proof. By Theorem 3.6, we have that $\mathrm{Chev}(K[t], (f)) \cap \mathcal{I}$ is normally generated by order p elements, and hence the same holds for $B := \ker(\varphi_f)$ (since the two groups are isomorphic because ϕ is injective). Since Λ has no non-trivial elements of order p the only possible morphism in $\hom_{\mathrm{Grp}}(B, \Lambda)$ is the trivial one. This implies that $H^1(B, \Lambda)$ is trivial, since the elements of the first non-Abelian cohomology group are precisely conjugacy classes of the elements of $\hom_{\mathrm{Grp}}(B, \Lambda)$. $\qquad\square$

The vanishing of first cohomology of congruence KMS complexes is summarized in the following corollary.

Corollary 3.8. *Let $\mathring{\Phi}$ by a spherical root system of rank $n \geq 4$ and K a finite field with $|K| \geq 5$ and $\mathrm{char}(K) = p \neq 2$. Let $f \in K[t]$ be an irreducible polynomial of degree at least 2. Let $X(\mathring{\Phi}, K, f)$ denote the corresponding congruence KMS complex as in Definition 2.17. Furthermore, let Λ be a group with no non-trivial elements of order p. Then*

$$H^1(X(\mathring{\Phi}, K, f), \Lambda) = 0.$$

Proof. This follows from Theorem 3.1 together with Proposition 3.4, Lemma 3.5 and Corollary 3.7.
$\qquad\square$

Remark 3.9. A similar argument, using Theorem 3.1 and Proposition 3.4, could be applied to other finite quotients of KMS groups used in the KMS complex construction as well, provided that one can show that the cohomology of the kernel of the quotient is trivial.

4 Non-Abelian cone functions

In the following section, we introduce non-Abelian cone functions, a tool to bound the coboundary expansion constants of strongly symmetric simplicial complexes. The definition is very similar to the work of Dinur and Dikstein in [DD24c] and we claim no originality here. Our contribution in this chapter is the description of new ways to construct cone functions with bounded radius in Section 4.2 and Section 4.3. These constructions are very similar to the ones for Abelian cone functions in [OVB25, Chapter 2.5, Chapter 3].

4.1 Definition and relation to coboundary expansion

Throughout, X is an n-dimensional simplicial complex.

Given two vertices u, v of X, a *path from u to v* is a sequence of k-tuple of vertices $(u_0; ...; u_k)$ where $u = u_0$ and $v = u_k$ such that for every $0 \leq i \leq k - 1$, $(u_i, u_{i+1}) \in X_{\mathrm{ord}}(1)$ (we use the ; symbol to distinguish between the notation of a path and that of an ordered simplex). The constant k is called the *length of the path*. Given two paths $P_0 = (u_0; ...; u_k), P_1 = (u_k; ...; u_m)$, we define $P_0 \circ P_1 = (u_0; ...; u_m)$. Also, given a path $P = (u_0; ...; u_k)$, we denote $P^{-1} = (u_k; ...; u_0)$.

A *loop around v* is a path from v to itself. We note that (v) is also considers as a loop and we will call it the *trivial loop around v*. Fix v_0 to be a vertex in X. We define the following symmetric relations between two loops P, P' around v_0:

- We denote $P \overset{(BT)}{\sim} P'$ if there exist vertices u, v and paths Q_0, Q_1 from v_0 to u such that $P = Q_0 \circ (u; v; u) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u) \circ Q_1^{-1}$ ((BT) stands for "backtracking").

- We denote $P \overset{(TR)}{\sim} P'$ if there is a triangle (u, v, w) and paths Q_0, Q_1 from v_0 to u and w correspondingly such that $P = Q_0 \circ (u; v; w) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u; w) \circ Q_1^{-1}$ ((TR) stands for "triangle").

Remark 4.1. Note that the relations above are not defined to be transitive. Indeed, in the sequel, it will be important to keep track of how many times the relation $\overset{(TR)}{\sim}$ is used to pass from one loop to another.

Definition 4.2. A 0-*cone* on X is a pair $\mathrm{Cone}_X = (v_0, \{P_u\}_{u \in X(0)})$ such that

(i) $v_0 \in X(0)$.

(ii) For every $u \in X(0), u \neq v_0$, P_u is a path from v_0 to u. For v_0, P_{v_0} is the trivial loop around v_0.

A 1-*cone* on X is a triple $\mathrm{Cone}_X = (v_0, \{P_u\}_{u \in X(0)}, \{T_{(u,v)}\}_{(u,v) \in X_{\mathrm{ord}}(1)})$ such that

(i) $v_0 \in X(0)$.

(ii) For every $u \in X(0), u \neq v_0$, P_u is a path from v_0 to u. For v_0, P_{v_0} is the trivial loop around v_0.

(iii) For every $(u, v) \in X_{\mathrm{ord}}(1)$, $T_{(u,v)}$ is a sequence of loops around v_0, $T_{(u,v)} = (P_0, P_1, \ldots, P_m)$ such that the following holds:

   (a) $P_0 = P_u \circ (u; v) \circ P_v^{-1}$.
   (b) For every $0 \leq i \leq m - 1$, either $P_i \overset{(TR)}{\sim} P_{i+1}$ or $P_i \overset{(BT)}{\sim} P_{i+1}$.
   (c) P_m is the trivial loop around v_0.

We call $T_{(u,v)} = (P_0, P_1, ..., P_m)$ as above a *contraction* and denote

$$|T_{(u,v)}| = |\{0 \leq i \leq m - 1 : P_i \overset{(TR)}{\sim} P_{i+1}\}|.$$

We further define the *radii of the cone* as follows:

(i) $\mathrm{Rad}_0(\mathrm{Cone}_X) = \sup_{u \in X(0)} \mathrm{length}(P_u)$.

(ii) $\mathrm{Rad}_1(\mathrm{Cone}_X) = \sup_{(u,v) \in X_{\mathrm{ord}}(1)} |T_{(u,v)}|$.

(iii) $\mathrm{Rad}(\mathrm{Cone}_X) = \begin{cases} \max\{\mathrm{Rad}_0(\mathrm{Cone}_X), \mathrm{Rad}_1(\mathrm{Cone}_X)\} & \text{if } \mathrm{Cone}_X \text{ is a 1-cone} \\ \mathrm{Rad}_0(\mathrm{Cone}_X) & \text{if } \mathrm{Cone}_X \text{ is a 0-cone} \end{cases}$.

Remark 4.3. The definition of $|T_{(u,v)}|$ in [DD24c] is slightly different than ours. Namely, they define another relation $\sim_1$ that is composed of applying $\overset{(TR)}{\sim}$ once and $\overset{(BT)}{\sim}$ as many times as needed and define $|T_{(u,v)}|$ as how many such relations $\sim_1$ are needed to bring P_0 to P_m such that P_m can be made trivial by only applying only the relations $\overset{(BT)}{\sim}$. We leave it to the reader to verify that both definitions coincide.

We denote by $\mathrm{Aut}(X)$ the group of simplicial automorphisms of X. The complex X is called *strongly symmetric* if $\mathrm{Aut}(X)$ acts transitively on $X(n)$, i.e., for every $\sigma_1, \sigma_2 \in X(n)$ there is $g \in \mathrm{Aut}(X)$ such that $g.\sigma_1 = \sigma_2$. In [DD24c], the following result was proven:

Lemma 4.4. *[DD24c, Lemma 1.6, Lemma 4.3] Let X be a pure n-dimensional finite simplicial complex, $n \geq 2$. Assume that there is a 1-cone on X such that $\mathrm{Rad}_i(\mathrm{Cone}_X) < \infty, i = 0, 1$ and that X is strongly symmetric, then for every group Λ,*

$$h^1_{\mathrm{cb}}(X, \Lambda) \geq \frac{1}{\binom{n+1}{3} \mathrm{Rad}_1(\mathrm{Cone}_X)}.$$

4.2 Cone functions of joins

We denote by $\sqcup$ the disjoint union of sets. Given two non-empty finite simplicial complexes Y_1 and Y_2 with vertex sets $V(Y_1)$ and $V(Y_2)$, the *join* of Y_1 and Y_2, denoted $Y_1 * Y_2$, is the simplicial complex with the vertex set $V(Y_1) \sqcup V(Y_2)$ and simplices $\sigma \sqcup \tau$ for every $\sigma \in Y_1$ and $\tau \in Y_2$. We note that $Y_1 * Y_2$ includes all the simplices of the form $\tau = \emptyset \sqcup \tau$ and $\sigma = \sigma \sqcup \emptyset$ for every $\sigma \in Y_1$ and $\tau \in Y_2$.

Proposition 4.5. *Let Y be a non-empty simplicial complex and w a vertex not in Y. Then there exists a 1-cone function $\mathrm{Cone}_{\{w\}*Y}$ with $\mathrm{Rad}_i(\mathrm{Cone}_{\{w\}*Y}) \leq 1$ for $i = 0, 1$.*

Proof. Set $X = \{w\} * Y$. First, assume that $\dim Y \geq 1$. We note that for every $(u, v) \in Y_{\mathrm{ord}}(1)$ it holds that $(w, u, v) \in X_{\mathrm{ord}}(2)$ and thus if P is the trivial loop around w, and P_u, P_v are two paths starting in w and ending in u, v respectively, then

$$P_u \circ (u; v) \circ P_v^{-1} \overset{(TR)}{\sim} P.$$

Thus, we define a function $\mathrm{Cone}_{\{w\}*Y} = (v_0, \{P_u\}_{u \in X(0)}, \{T_{(u,v)}\}_{(u,v) \in X_{\mathrm{ord}}(1)})$ as follows:

- $v_0 = w$.

- For every $u \in X(0)$, $P_u = (w; u)$.

- For every $(u, v) \in X_{\mathrm{ord}}(1)$, $T_{u,v} = (P_0 = P_u \circ (u; v) \circ P_v^{-1}, P_1 = P_w)$.

It follows that $\mathrm{Rad}_i(\mathrm{Cone}_{\{w\}*Y}) = 1, i = 0, 1$ as needed.

If $\dim Y = 0$ then $\dim X = 1$. But we can still define a 1-cone function of radius 1: The v_0 and P_u are defined as before. The only edges in $X_{\mathrm{ord}}(1)$ are of the form (w, u) (or (u, w)) with $u \in Y(0)$. Thus $T_{(w,u)}$ consists of $P_0 = (w) \circ (w, u) \circ (u, w) \sim_{(BT)} P_1 = (w)$ (and $T_{(u,w)}$ of $P_0 = (w, u) \circ (u, w) \circ (w) \sim_{(BT)} (w)$). $\square$

Remark 4.6. We note that in the above construction, the cone radius is still ≤ 1, even if Y has no edges or if Y is disconnected. If Y is empty, we can still say that $\{w\} * \emptyset$ has a 0-cone function, since it will be a connected graph consisting of one single vertex.

Lemma 4.7. *Let Y_1, Y_2 be non-empty, finite simplicial complexes. Then $Y_1 * Y_2$ admits a 0-cone function with $\mathrm{Rad}_0(\mathrm{Cone}_{Y_1*Y_2}) = 2$.*

Proof. We have that the 1-skeleton of $Y_1 * Y_2$ contains the complete bipartite graph on $Y_1(0) \sqcup Y_2(0)$ as a subgraph. In particular, every two vertices are connected by a path of length at most 2. $\square$

Proposition 4.8. *Let Y_1 be a n-dimension simplicial complex, $n \geq 1$, with 0-cone function* $\mathrm{Cone}_{Y_1} = (v_0^1, (P_u^1)_{u \in Y_1(0)})$. *Let Y_2 be a non-empty, finite simplicial complex. Then $Y_1 * Y_2$ admits a 1-cone function* $\mathrm{Cone}_{Y_1 * Y_2}$ *with* $\mathrm{Rad}_0(\mathrm{Cone}_{Y_1 * Y_2}) = \mathrm{Rad}_0(\mathrm{Cone}_{Y_1})$ *and* $\mathrm{Rad}_1(\mathrm{Cone}_{Y_1 * Y_2}) \leq 2 \mathrm{Rad}_0(\mathrm{Cone}_{Y_1}) + 1$.

Proof. We want to define a 1-cone $(v_0, (P_u)_{u \in Y_1(0) \sqcup Y_2(0)}, (T_{(u,v)})_{(u,v) \in (Y_1 * Y_2)_{\mathrm{ord}}(1)})$ for $Y_1 * Y_2$. First, note that $(Y_1 * Y_2)_{\mathrm{ord}}(1) = Y_{1\mathrm{ord}}(1) \sqcup Y_{2\mathrm{ord}}(1) \sqcup \{(u,v) \mid u \in Y_i, v \in Y_j, \{i,j\} = \{1,2\}\}$.

- We set $v_0 = v_0^1$.

- For $u \in Y_1(0)$, we set $P_u = P_u^1$.

- For $u \in Y_2(0)$, we set $P_u = (v_0, u)$.

- For $(u,v) \in Y_{2\mathrm{ord}}(1)$, we set $T_{(u,v)} = (P_0 = (v_0; u; v; v_0), P_1 = (v_0; v; v_0), P_2 = (v_0))$ with $P_0 \sim_{(TR)} P_1$ and $P_1 \sim_{(BT)} P_2$.

- Fix $u \in Y_1(0), v \in Y_2(0)$ and let $P_u = (v_0; u_1; \ldots; u_n; u)$. Then we define $T_{(u,v)}$ as follows, see also Figure 2.

$$P_0 = (v_0; u_1; \ldots; u_n) \circ (u_n, u) \circ (u, v) \circ (v, v_0)$$
$$\wr_{(TR)} \text{ with triangle } (u_n, u, v)$$
$$P_1 = (v_0; u_1; \ldots; u_n) \circ (u_n, v) \circ (v, v_0)$$
$$\wr_{(TR)}$$
$$\vdots$$
$$\wr_{(TR)}$$
$$P_n = (v_0, u_1) \circ (u_1, v) \circ (v, v_0)$$
$$\wr_{(TR)} \text{ with triangle } (v_0, u_1, v)$$
$$(v_0, v) \circ (v, v_0)$$
$$\wr_{(BT)}$$
$$(v_0)$$

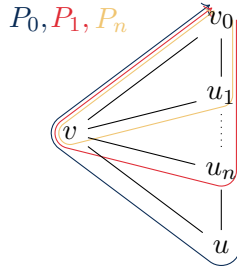Observe that $|T_{(u,v)}| = \mathrm{length}(P_u)$.

Figure 2: Visualization of the paths in $T_{(u,v)}$ for $u \in Y_1(0), v \in Y_2(0)$.

- Lastly, fix $(u,v) \in Y_{1 \mathrm{ord}}(1)$ and denote $P_u = (v_0; u_1; \ldots; u_n; u), P_v = (v_0; v_1; \ldots; v_n; v)$. Then we define $T_{(u,v)}$ as follows, similar to the above case, but now doing the contractions on "both sides" of the path. We also need to fix an arbitrary vertex $x \in Y_2(0)$. See also Figure 3.

$$P_0 = (v_0; u_1; \ldots; u_n) \circ (u_n, u) \circ (u, v) \circ (v, v_n) \circ (v_n; \ldots; v_1; v_0)$$

$$\wr_{(TR)} \text{ with triangle } (u, x, v)$$

$$P_1 = (v_0; u_1; \ldots; u_n) \circ (u_n, u) \circ (u, x) \circ (x, v) \circ (v, v_n) \circ (v_n; \ldots; v_1; v_0)$$

$$\wr_{(TR)} \text{ with triangle } (u_n, u, x) \text{ and } (v, v_n, x)$$

$$P_3 = (v_0; u_1; \ldots; u_n) \circ (u_n, x) \circ (x, v_n)(v_n; \ldots; v_0)$$

$$\vdots$$

$$(v_0, u_1) \circ (u_1, x) \circ (x, v_1) \circ (v_1, v_0)$$

$$\wr_{(TR)} \text{ with triangle } (v_0, u_1, x) \text{ and } (x, v_1, v_0)$$

$$(v_0, x) \circ (x, v_0)$$

$$\wr_{(BT)}$$

$$(v_0)$$

Observe that the paths P_u, P_v don't necessary have to be of the same length. In that case,
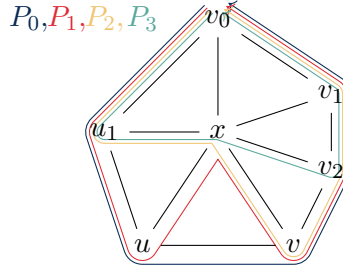

Figure 3: Visualization of the paths in $T_{(u,v)}$ for $(u,v) \in Y_{1 \mathrm{ord}}(1)$ (for an example with $\mathrm{length}(P_u) = 2, \mathrm{length}(P_v) = 3$).

the induction on one side halts earlier then on the other side. Counting the total number of (TR) operations made, we have $|T_{(u,v)}| = \mathrm{length}(P_u) + \mathrm{length}(P_v) + 1$.

$\square$

Proposition 4.9. *Let Y_1, Y_2 be finite, non-empty, simplicial complexes. Assume that there is a 1-cone function Cone_{Y_1} with finite cone radius. Then there is a 1-cone function $\mathrm{Cone}_{Y_1 * Y_2}$ such that,*

$$\mathrm{Rad}_0(\mathrm{Cone}_{Y_1 * Y_2}) = \mathrm{Rad}_0(Y_1)$$

$$\mathrm{Rad}_1(\mathrm{Cone}_{Y_1 * Y_2}) = \max\{\mathrm{Rad}_0(Y_1), \mathrm{Rad}_1(Y_1), 1\}.$$

Proof. Recall that $(Y_1 * Y_2)(0) = Y_1(0) \sqcup Y_2(0)$ and that any simplex of either Y_i can be seen as a simplex in the join as well. In particular, $(Y_1 * Y_2)_{\mathrm{ord}}(1) = Y_{1 \mathrm{ord}}(1) \sqcup Y_{2 \mathrm{ord}}(1) \sqcup \{(u,v) \mid u \in Y_i(0), v \in Y_j(0), i, j \in \{1, 2\}, i \neq j\}$. Let $\mathrm{Cone}_{Y_1} = (v_0^1, (P_u^1)_{u \in Y_1(0)}, (T_{(u,v)}^1)_{(u,v) \in Y_{1 \mathrm{ord}}(1)})$. We will define $\mathrm{Cone}_{Y_1 * Y_2} = (v_0, \{P_u\}_{u \in Y_1 * Y_2(0)}, \{T_{(u,v)}\}_{(u,v) \in (Y_1 * Y_2)_{\mathrm{ord}}(1)})$ as follows.

- We set $v_0 = v_0^1$.

- For $u \in Y_1(0)$, set $P_u = P_u^1$.

- For $v \in Y_2(0)$, set $P_v = (v_0, v)$. (Note that this is well defined since $v_0 \in Y_1$ is connected by an edge to any vertex of Y_2.)

- For $(u, v) \in Y_{1\mathrm{ord}}(1)$, set $T_{(u,v)} = T_{(u,v)}^1$.

- For $(u, v) \in Y_{2\mathrm{ord}}(1)$, set $T_{(u,v)} = (P_0 = (v_0, u) \circ (u, v) \circ (v, v_0), P_1 = (v_0, v) \circ (v, v_0), P_2 = (v_0))$. Note that $P_0 \sim_{(TR)} P_1$ and $P_1 \sim_{(BT)} P_2$.
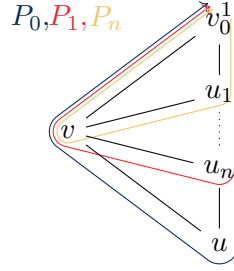

Figure 4: Visualization of the paths in $T_{(u,v)}$ for $u \in Y_1(0), v \in Y_2(0)$.

- Lastly, we define the most involved case, where $u \in Y_1(0), v \in Y_2(0)$. We have $P_0 = P_u \circ (u, v) \circ (v, v_0)$. We give names to the vertices in P_u: $P_u = (v_0; u_1; u_2; \ldots; u_n; u)$. Note that each u_i is also connected to v by an edge. Next we define $P_1 = (v_0; u_1; \ldots; u_n) \circ (u_n, v) \circ (v, v_0)$ and observe that $P_0 \sim_{(TR)} P_1$ via the triangle (u_n, u, v). We continue this process iteratively, see also Figure 4: $P_i = (v_0; u_1; \ldots; u_{n-i+1}) \circ (u_{n-i+1}, v) \circ (v, v_0)$, hence $P_{i-1} \sim_{(TR)} P_i$ via the triangle $(u_{n-i+1}, u_{n-i+2}, v)$ for $i = 1, \ldots, n$. Then $P_n = (v_0; u_1; v; v_0)$, and we finish the contraction via $P_{n+1} = (v_0; v; v_0), P_{n+2} = (v_0)$ with $P_n \sim_{(TR)} P_{n+1}$ and $P_{n+1} \sim_{(BT)} P_{n+2}$.

  In particular, we have used (TR) once for every edge in P_u, hence $|T_{(u,v)}| = \mathrm{length}(P_u)$.

  The definition of $T_{(v,u)}$ is analogues, just with reversed direction of the paths.

$\square$

To avoid having to make case distinctions between cone functions for 1-dimensional complexes and complexes with dimension ≥ 2, we introduce the following notation.

Definition 4.10. Let X be a finite simplicial complex. We say that X has a non-Abelian cone function (NAC) if

- $X \neq \emptyset$,

- if $\dim X = 1$ there exists a 0-cone function for X,

- if $\dim X \geq 2$ there exists a 1-cone function for X.

Furthermore, we define the cone radius of X, denoted $\mathrm{Rad}(X)$ as

- if $\dim X = 0, X \neq \emptyset$ then $\mathrm{Rad}(X) = 1$,

- if $\dim X \geq 1$ and X has a NAC then $\mathrm{Rad}(X) = \min\{\mathrm{Rad}(\mathrm{Cone}_X) \mid \mathrm{Cone}_X$ a 1-cone function of $X\}$,

- if X does not admit a NAC, we set $\mathrm{Rad}(X) = \infty$.

With this notation, we can summarize the results of this chapter. By summarizing the different cases, we loose some sharpness of the bounds on the cone radius.

Corollary 4.11. *Let X, Y be two finite simplicial complexes. If both X and Y have a NAC then $X * Y$ has a NAC and $\mathrm{Rad}(X * Y) \leq 2 \max\{\mathrm{Rad}(X), \mathrm{Rad}(Y)\} + 1$.*

Proof. If $\dim X = \dim Y = 0$ then $\dim X * Y = 1$ and it has a 0-cone function by Lemma 4.7. If $\dim X = 1$ then the assumption implies that X has a 0-cone function. Thus Proposition 4.8 gives the desired result. If $\dim X \geq 2$, then by assumption there exists a 1-cone function for X. Hence Proposition 4.9 gives the desired result. $\qquad\square$

4.3 Constructing cone functions by adding vertices

The idea is to construct a cone function for a given complex X, by starting with a given subcomplex for which we have a cone function and adding a set of vertices which is "well-behaved". The exact formalism of this idea is given in Theorem 4.13 that can be thought of as a special version of the Mayer-Vietoris theorem for cone functions.

Definition 4.12. For a complex X, a subcomplex $X' \subseteq X$ is called full, if for every $v_0, \dots, v_k \in X'$, if $\{v_0, \dots, v_k\} \in X$, then $\{v_0, \dots, v_k\} \in X'$.

Theorem 4.13. *Let X be an n-dimensional simplicial complex, $n \geq 2$, and let X' be a full subcomplex. Assume there is a 1-cone function $\mathrm{Cone}_{X'}$ and constants $R'_j \in \mathbb{N}, j = 0, 1$ such that for every $j = 0, 1$, $\mathrm{Rad}_j(\mathrm{Cone}_{X'}) \leq R'_j$. Also let $W \subseteq X(0)$ be a set of vertices such that the following holds:*

(i) $W \cap X' = \emptyset$.

(ii) For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.

(iii) For every $w \in W$, $\mathrm{lk}_X(w) \cap X'$ is a non-empty simplicial complex.

(iv) There are 0-cone functions $\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}$ for all $w \in W$ and a constant $R'' \in \mathbb{N}$ such that for every $w \in W$, $\mathrm{Rad}_0(\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}) \leq R''$.

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then there is a 1-cone function $\mathrm{Cone}_{X' \cup W}$ such that $\mathrm{Rad}_0(\mathrm{Cone}_{X' \cup W}) \leq R'_0 + 1, \mathrm{Rad}_1(\mathrm{Cone}_{X' \cup W}) \leq R''(R'_1 + 1)$.

Proof. First, we fix some notation: for each $w \in W$ let $L_w := \mathrm{lk}_X(w) \cap X'$ and

$$\mathrm{Cone}_{X'} = (v'_0, (P'_u)_{u \in X'(0)}, (T'_{(u,v)})_{(u,v) \in X'_{\mathrm{ord}}(1)})$$
$$\mathrm{Cone}_{L_w} = (v^w_0, (P^w_u)_{u \in L_w(0)}).$$

Observe that $L_w(0)$ are those vertices of X' which are neighbors of w, that $(X' \cup W)(0) = X'(0) \cup W$ and that edges in $X' \cup W$ are all edges of X' and $\{\{u, w\} \mid w \in W, u \in L_w\}$.

We define $\mathrm{Cone}_{X' \cup W} = (v_0, (P_u)_{u \in X'(0) \cup W}, (T_{(u,v)})_{(u,v) \in (X' \cup W)_{\mathrm{ord}}(1)})$ as follows.

- We set $v_0 = v_0' \in X'$.

- For $u \in X'(0)$, we set $P_u = P_u'$.

- For $w \in W$, we set $P_w = P_{v_0^w}' \circ (v_0^w, w)$.

- For $(u, v) \in X'_{\mathrm{ord}}(1)$, we set $T_{(u,v)} = T'_{(u,v)}$.

- Lastly, we define the most involved case. Let $w \in W, u \in L_w$ (hence $(u, w) \in (X' \cup W)_{\mathrm{ord}}(1)$). Then we define $T_{(u,w)}$ as follows (and $T_{(w,u)}$ similarly by reversing all paths). We have $P_0 = P_u' \circ (u, w) \circ (w, v_0^w) \circ (P_{v_0^w}')^{-1}$. We give names to the vertices in $P_u^w = (v_0^w; u_1; \ldots; u_n; u)$. We get the following series of contractions, see also Figure 5.
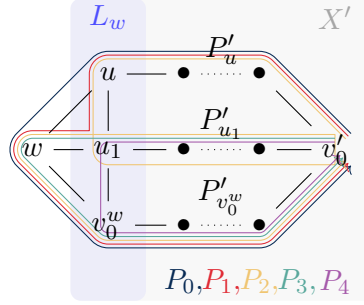

Figure 5: Visualization of $T_{(u,w)}$ shown in the case $\mathrm{length}(P_u^w) = 2$.

$$P_0 = P_u \circ (u, w) \circ P_w^{-1}$$

$$\wr_{(TR)} \text{ with triangle } (u, w, u_n)$$

$$P_u \circ (u, u_n) \circ (u_n, w) \circ P_w^{-1}$$

$$\wr_{(BT)} \text{ as often as necessary}$$

$$P_u \circ (u, u_n) \circ P_{u_n}^{-1} \circ P_{u_n} \circ (u_n, w) \circ P_w^{-1}$$

$$\wr \text{ apply sequence of relations from } T'_{(u, u_n)}$$

$$(v_0) \circ P_{u_n} \circ (u_n, w) \circ P_w^{-1}$$

$$\wr_{(TR)} \text{ with triangle } (u_n, w, u_{n-1})$$

$$P_{u_n} \circ (u_n, u_{n-1}) \circ (u_{n-1}, w) \circ P_w^{-1}$$

$$\wr_{(BT)} \text{ as often as necessary}$$

$$P_{u_n} \circ (u_n, u_{n-1}) \circ P_{u_{n-1}}^{-1} \circ P_{u_{n-1}} \circ (u_{n-1}, w) \circ P_w^{-1}$$

$$\wr \text{ apply sequence of relations from } T'_{(u_n, u_{n-1})}$$

$$(v_0) \circ P_{u_{n-1}} \circ (u_{n-1}, w) \circ P_w^{-1}$$

$$\vdots$$

$$P_{u_1} \circ (u_1, w) \circ P_w^{-1}$$

$$\wr_{(TR)} \text{ with triangle } (u_1, w, v_0^w)$$

$$P_{u_1} \circ (u_1, v_0^w) \circ (v_0^w, w) \circ P_w^{-1}$$

$$\wr_{(BT)} \text{ as often as necessary}$$

$$P_{u_1} \circ (u_1, v_0^w) \circ P_{v_0^w}^{-1} P_{v_0^w} (v_0^w, w) P_w^{-1}$$

$$\wr \text{ apply sequence of relations from } T'_{(u_1, v_0^w)}$$

$$(v_0) \circ P_{v_0^w} \circ (v_0^w, w) \circ (w, v_0^w) P_{v_0^w}^{-1}$$

$$\wr_{(BT)} \text{ as often as necessary}$$

$$(v_0)$$

In total, for each edge in P_u^w we have used at most $R_1' + 1$ triangle relations. Hence in total $|T_{(u,w)}| \le R''(R_1' + 1)$.

$\square$

We need a slightly adapted version for 1-dimensional complexes.

Lemma 4.14. *Let X be a one-dimensional simplicial complex, and let X' be a full subcomplex. Assume there is a 0-cone function $\mathrm{Cone}_{X'}$ and constant $R' \in \mathbb{N}$ such that $\mathrm{Rad}_0(\mathrm{Cone}_{X'}) \le R'$. Also let $W \subseteq X(0)$ be a set of vertices such that the following holds:*

(i) $W \cap X' = \emptyset$.

(ii) For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.

(iii) For every $w \in W$, $\mathrm{lk}_X(w) \cap X'$ is non-empty.

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then there is an 0-cone function $\mathrm{Cone}_{X' \cup W}$ such that $\mathrm{Rad}_0(\mathrm{Cone}_{X' \cup W}) \leq R' + 1$

Proof. Let $\mathrm{Cone}_{X'} = (v'_0, (P'_u)_{u \in X'(0)})$. We define $\mathrm{Cone}_{X' \cup W} = (v_0, (P_u)_{u \in X'(0) \cup W})$ as follows.

- Set $v_0 = v'_0$.

- For $u \in X'(0)$, set $P_u = P'_u$.

- For $w \in W$, pick a vertex $v \in \mathrm{lk}_X(w) \cap X'$, and set $P_w = P_v \circ (v, w)$.

$\square$

We can summarize both results in the following corollary. Note here that a 0-dimensional complex has a 0-cone function if and only if it contains exactly one vertex.

Corollary 4.15. *Let X be a finite simplicial complex of dimension at least 1 and let X' be a full subcomplex of dimension at least 1. Assume that X' has a NAC and $\mathrm{Rad}(X') \leq R'$ for some constant $R' \in \mathbb{N}$. Let $W \subseteq X(0)$ be a set of vertices and $R'' \in \mathbb{N}$ such that*

(i) $W \cap X' = \emptyset$.

(ii) For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.

(iii) For every $w \in W$, $\mathrm{lk}_X(w) \cap X'$ has a 0-cone function $\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}$ with $\mathrm{Rad}_0(\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}) \leq R''$.

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then $X' \cup W$ has a NAC and $\mathrm{Rad}(X' \cup W) \leq R''(R' + 1)$.

5 Cosystolic expansion with non-Abelian coefficients

In this section, we prove the following theorem which is the last key step in proving that classical congruence KMS complexes give rise to infinite families of bounded degree 1-coboundary expanders.

Theorem 5.1. *Let $n \geq 3$ be an integer. There exists an $\varepsilon > 0$ such that for all prime powers $q > 2^{2n-1}$, every n-dimensional, n-classical KMS complex X over $\mathbb{F}_q$ and every group Λ all proper links of X are (Λ, ε) 1-coboundary expanders.*

Together with the local-to-global machinery (Theorem 2.2) and the results on local-spectral expansion (Theorem 2.16) this theorem gives rise to the following corollary.

Corollary 5.2. *Let $n \geq 3$ be an integer. There exists $\varepsilon > 0, q_0 \in \mathbb{N}$ such that for all prime powers $q \geq q_0$, every n-dimensional, n-classical KMS complex X over $\mathbb{F}_q$ and every group Λ, we have $h^1_{cs}(X, \Lambda) \geq \varepsilon$.*

Combining this with the result from Corollary 3.8, we can then deduce the main result of this work.

Corollary 5.3. *Let $n \geq 3$ and let $\mathring{\Phi}$ be root system of type A_n, B_n, C_n, or D_n then there exists $q_0 \in \mathbb{N}, \varepsilon > 0$ such that*

- *for every finite field K with $|K| \geq q_0$ and $\mathrm{char}(K) =: p \neq 2$,*

- *for every group Λ with no non-trivial elements of order p (e.g., if Λ is finite and $|\Lambda| < p$),*

- *for every irreducible polynomial $f \in K[t]$ of degree at least 2*

the congruence KMS complex $X(\mathring{\Phi}, K, f)$ is a (Λ, ε) 1-coboundary expander.

In particular, given a finite group Λ, it holds for any sufficiently large finite field K with $\mathrm{char}(K) > |\Lambda|$ and any infinite family of irreducible polynomials $f_s \in K[t]$ of degree at least 2 with $\lim_{s \to \infty} \deg(f_s) = \infty$ that the family $\{X(\mathring{\Phi}, K, f_s)\}_{s \in \mathbb{N}}$ is a family of 1-coboundary expanders with Λ coefficients.

Proof. Let $X_f = X(\mathring{\Phi}, K, f), (\varphi_f(U_{I \setminus \{i\}}))_{i \in I}$. First, Theorem 2.2, Theorem 2.16 and Theorem 5.1 imply that $h^1_{\mathrm{cs}}(X_f) > \varepsilon$ for some $\varepsilon > 0$ only depending on n, A, q_0. Then Corollary 3.8 shows that $h^1_{\mathrm{cs}}(X_f) = h^1_{\mathrm{cb}}(X_f)$. The bound on $h^0(X_f)$ follows from the fact that X_f is a λ-local spectral expander for some λ depending only on A, n, q_0 (Theorem 2.16) which implies that the 1-skeleton is a λ-expander graph. $\square$

Remark 5.4. Note that the results of Theorem 5.1 hold for all classical KMS complexes, while the result of Corollary 3.8 hold for all congruence KMS complexes. To get Corollary 5.3, both results need to hold, thus we consider the classical congruence KMS complexes.

Theorem 5.1 is an adaption of [OVB25, Theorem 1.1] to the setting with non-Abelian coefficients. The proof structure is very similar, and we just adapt it to the non-Abelian setting when necessary.

The idea of the proof is as follows: First, one observes that proper links of KMS complexes are isomorphic to joins of certain subcomplexes of spherical buildings, called opposite complexes. To further elaborate on this point:

- Each proper link has an associate subdiagram of the Dynkin diagram of the underlying GCM.

- Each connected component of the subdiagram corresponds to an irreducible spherical building.

- Each spherical building admits an opposite relation between its simplices, and after fixing one "fundamental chamber" the set of simplices opposite this chamber is called the opposite complex.

- A proper link of the KMS complex is then isomorphic to the join of the opposite complexes inside the buildings corresponding to the connected components of the Dynkin diagram. This was shown in [GdPVB25, Chapter 3] and we describe more details in Appendix A.1.

Second, given two complexes with a 1-cone function, we can construct a 1-cone function for their join as well. Hence it suffices to check that all opposite complexes of irreducible spherical buildings admit a NAC.

Third, for all "classical" buildings, i.e. those of type A_n, C_n, D_n, we construct a NAC for their opposite complex. This is done by using an inductive argument on a class of complexes which contains the desired opposite complexes. This argument was first used by Abramenko in [Abr96]

to show $(n-1)$-sphericallity of the opposite complexes and was adapted to show the existence of Abelian cone functions in [OVB25]. Below, we prove the non-Abelian version of the induction argument. The proof of the fact that the relevant classes satisfy the requirements of the theorem can be found in the appendix for completeness, but it almost directly follows from the proof for the Abelian case.

We note that the opposite complexes of buildings of type D_n require a bit more technical work and therefore are dealt with in the second part of this section.

5.1 Non-Abelian cone functions for opposite complexes of type A_n and C_n

The existence of non-Abelian cone functions of opposite complexes of buildings of type A_n and C_n will follow from the following theorem, which is a quantitative version of [Abr96, Lemma 22], see also [OVB25, Theorem 6.1].

Theorem 5.5. *Let* $\mathbf{C}$ *be a class of non-empty simplicial complexes. Assume that for any* $n \in \mathbb{N}, n \geq 2$ *there exists* $\ell_n \in \mathbb{N}$ *such that for all* $\kappa \in \mathbf{C}, \dim \kappa = n$, *there exists a filtration* $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ *of* κ *of length* $\ell \leq \ell_n$ *such that the following is satisfied (set* $V_i := \{vertices\ of\ \kappa_i\} \setminus \{vertices\ of\ \kappa_{i-1}\}$):

   (i) There is an 1-cone function Cone_{κ_0} *such that for* $j = 0, 1$,

$$\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_0}) \leq f(n),$$

      where $f : \mathbb{N} \to \mathbb{R}$ *is a function depending only on* $\mathbf{C}$.

   (ii) κ_i *is a full subcomplex of* κ *and for all vertices* $w_1, w_2 \in V_i$ *we have* $\{w_1, w_2\} \notin \kappa$, *for all* $0 \leq i \leq \ell$.

   (iii) $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ *is either in* $\mathbf{C}$ *or can be written as the join of two elements from* $\mathbf{C}$ *for any* $w \in V_i, 1 \leq i \leq \ell$.

Furthermore, we assume that there exists $\ell_1 \in \mathbb{N}$ *such that for any* $\kappa \in \mathbf{C}$ *with* $\dim(\kappa) = 1$, *there exists a filtration* $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ *of* κ *of length* $\ell \leq \ell_1$ *such that the following is satisfied (set* $V_i := \{vertices\ of\ \kappa_i\} \setminus \{vertices\ of\ \kappa_{i-1}\}$):

   (i) There is an 0-cone function Cone_{κ_0} *such that*

$$\mathrm{Rad}_0(\mathrm{Cone}_{\kappa_0}) \leq f(1),$$

   (ii) κ_i *is a full subcomplex of* κ *and for all vertices* $w_1, w_2 \in V_i$ *we have* $\{w_1, w_2\} \notin \kappa$, *for all* $0 \leq i \leq \ell$,

   (iii) $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ *is non-empty for all* $w \in V_i, 1 \leq i \leq \ell$.

Then for every $n \in \mathbb{N}, n \geq 2$ *there exists a constant* $\mathcal{R}(n)$ *such that for every* $\kappa \in \mathbf{C}$ *with* $\dim \kappa = n$ *we have:* κ *has a 1-cone function* Cone_κ *such that*

$$\mathrm{Rad}(\mathrm{Cone}_\kappa) \leq \mathcal{R}(n).$$

Furthermore, every $\kappa \in \mathbf{C}$ *with* $\dim(\kappa) = 1$ *admits a 0-cone function with* $\mathrm{Rad}_0(\mathrm{Cone}_\kappa) \leq f(1) + \ell_1$.

We can describe $\mathcal{R}(n)$ recursively. We have $\mathcal{R}(0) = 1$. For $n \geq 1$, assume $\mathcal{R}(k)$ is known for $k < n$. Set

$$S(n) = \max\{2, \mathcal{R}(n-1)\}.$$

Then

$$\mathcal{R}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j.$$

Proof. We will prove the result by induction on the dimension n.

For $n = 1$, we can apply Lemma 4.14 inductively to each step of the filtration to get a 0-come function Cone_κ with $\text{Rad}_0(\text{Cone}_\kappa) \leq f(1) + \ell$.

Fix $n \geq 2$ and $\kappa \in \mathbf{C}, \dim(\kappa) = n$ with filtration $\kappa_0 \subset \ldots \subset \kappa_\ell = \kappa$ as in the requirements. We will prove by induction on $0 \leq i \leq \ell$, that there is an 1-cone function Cone_{κ_i} such that

$$\text{Rad}(\text{Cone}_{\kappa_i}) \leq \mathcal{R}^{(i)}(n)$$

where $\mathcal{R}^{(i)}(n)$ is given inductively by

$$\mathcal{R}^{(0)}(n) = f(n), \qquad \mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n) + 1) = S(n)^i f(n) + \sum_{j=1}^{i} S(n)^j.$$

For $i = 0$ this follows from Assumption 1. We proceed by induction on i. Fix $1 \leq i \leq \ell$ and assume there exists a constant $\mathcal{R}^{(i-1)}(n)$ and a 1-cone function $\text{Cone}_{\kappa_{i-1}}$ such that $\text{Rad}(\text{Cone}_{\kappa_{i-1}}) \leq \mathcal{R}^{(i-1)}(n)$. We want to apply Theorem 4.13 to the following setting:

$$X = \kappa, X' = \kappa_{i-1}, \ \text{Cone}_{X'} = \text{Cone}_{\kappa_{i-1}}, \ R' = \mathcal{R}^{(i-1)}(n), \ W = V_i = \kappa_i(0) \setminus \kappa_{i-1}(0).$$

We check that the conditions of the theorem are satisfied.

(i) $W \cap X' = \emptyset$ by definition.

(ii) For every $w_1, w_2 \in W : \{w_1, w_2\} \notin \kappa_i \subset \kappa$ by assumption.

(iii) For every $w \in W, \text{lk}_X(w) \cap X'$ is a non-empty simplicial complex, since by Assumption 3 we have $\text{lk}_X(w) \cap X' \in \mathbf{C}$ or a join of two complexes in $\mathbf{C}$.

(iv) By Assumption 3, $\text{lk}_X(w) \cap X'$ is either in $\mathbf{C}$ or is a join of two elements of $\mathbf{C}$. In the first case we use the induction hypothesis and get an 0-cone function $\text{Cone}_{\text{lk}_X(w) \cap X'}$ such that

$$\text{Rad}_0(\text{Cone}_{\text{lk}_X(w) \cap X'}) \leq \mathcal{R}(n-1).$$

(Note that any 1-cone function also gives rise to a 0-cone function trivially.)

In the second case, we can write $\text{lk}_X(w) \cap X' = A * B$ for some $A, B \in \mathbf{C}$. Thus by Lemma 4.7, there exists a 0-cone function Cone_{A*B} with $\text{Rad}_0(\text{Cone}_{A*B}) \leq 2$.

Hence we can set $R'' = \max\{2, \mathcal{R}(n-1)\} = S(n)$.

Applying Theorem 4.13 we get a 1-cone function $\text{Cone}_{X' \cup W}$ such that

$$\text{Rad}(\text{Cone}_{X' \cup W}) \leq R''(R' + 1)$$

which in the original formulation means

$$\text{Rad}(\text{Cone}_{\kappa_i}) \leq S(n)(\mathcal{R}^{(i-1)}(n) + 1).$$

Thus we have $\mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n) + 1)$ and solving the recursion gives the second expression of $\mathcal{R}^{(i)}(n)$.

Now, since $\kappa_\ell = \kappa$, we get a 1-cone function Cone_κ with $\text{Rad}(\text{Cone}_\kappa) \leq \mathcal{R}^{(\ell)}(n)$. We would like to set $\mathcal{R}(n) = \mathcal{R}^{(\ell)}(n)$. But the bound has to hold for all possible $\kappa \in \mathbf{C}$ of dimension n. Since $\mathcal{R}^{(i)}(n)$ is increasing in i, we take $\mathcal{R}(n) = \mathcal{R}^{(\ell_n)}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j$.

$\square$

Remark 5.6. Assume that a class of simplicial complexes $\mathbf{C}$ satisfies the assumptions of the general induction argument in the Abelian case [OVB25, Theorem 6.1]. To show that $\mathbf{C}$ satisfies the conditions of Theorem 5.5, it suffices to show that the κ_0 of the filtration of each complex from the Abelian version of the theorem also admits a non-Abelian 1-cone function (or 0-cone function in the case where it is one-dimensional). The fact that this holds in the case of the classes containing the A_n and C_n opposite complexes follows, since the cone functions for the κ_0 are constructed using joins and the "adding vertices" construction, which both also work in the non-Abelian case.

We get the following corollaries, where we denote for a building Δ and $a \in \Delta$ by $\Delta^\circ(a)$ the complex opposite of a. More details on the opposite relation are explained in Appendix A.1. The idea of the proofs of the corollaries was explained in Remark 5.6, for completeness, we provide the details in Appendix B and Appendix C, .

Corollary 5.7 (Corollary B.4)**.** *For every $n \in \mathbb{N}, n \geq 2$ there is a constant $\mathcal{R}(n-1)$ such that for every building Δ of type $A_n, n \geq 2$ (which is a building of dimension $n - 1$) over a field K with $|K| \geq 2^{n-1}$ and $a \in \Delta$ a simplex, there exists a NAC $\text{Cone}_{\Delta^0(a)}$ of $\Delta^0(a)$ such that*

$$\text{Rad}(\text{Cone}_{\Delta^0(a)}) \leq \mathcal{R}(n-1).$$

Corollary 5.8 (Corollary C.2)**.** *For every $n \in \mathbb{N}, n \geq 2$ there is a constant $\mathcal{R}(n-1)$ such that for every building Δ of type $C_n, n \geq 2$ (which is a building of dimension $n - 1$) over a field K with $|K| \geq 2^{2n-1}$ and $a \in \Delta$ a simplex, there exists a NAC $\text{Cone}_{\Delta^0(a)}$ of $\Delta^0(a)$ such that*

$$\text{Rad}(\text{Cone}_{\Delta^0(a)}) \leq \mathcal{R}(n-1).$$

5.2 Non-Abelian cone functions for opposite complexes of type D_n

The difficulties in this case originate from the incidence geometric description of buildings of type D_n. In [Abr96], Abramenko works with a weak C_n type building and then describes the connection between it and the thick D_n building (basically, one is a subdivision of the other). Thus, we first need to prove that a 1-cone function for the opposite complexes of a weak C_n building gives rise to a 1-cone function for the corresponding thick D_n case, analogues to [OVB25, Proposition 6.10]. To do so, we briefly recall the geometric set up, more details can be found in the appendix.

5.2.1 Buildings of type D_n - geometric set up

For this section, let (V, Q, f) be a pseudo-quadratic space but in the specific case where K is a field, $\sigma = \mathrm{id}_K, \varepsilon = 1, \Lambda = 0, V$ a K-vector space of dimension $m = 2n \geq 4$, Q an ordinary quadratic form and f the corresponding symmetric bilinear form. We further assume that V is non-degenerate and of Witt index n.

We set $X = X(V) = \{0 < U < V \mid U \text{ is totally isotropic}\}$. Then $\Delta = \mathrm{Flag}\, X$ is a weak C_n building, in particular it is not thick, since each totally isotropic subspace of dimension $n - 1$ is contained in exactly two totally isotropic subspaces of dimension n. Furthermore, the space $Y = \{U \in X \mid \dim U = n\}$ is partitioned into two sets $Y = Y_1 \sqcup Y_2$ such that for $U_1, U_2 \in Y_i$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z}$ for $i = 1, 2$ and for $U_1 \in Y_1, U_2 \in Y_2$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z} + 1$. In particular, if for $U_1, U_2 \in Y$ we have $\dim(U_1 \cap U_2) = n - 1$ then U_1 and U_2 are in different sets of the partition.

To obtain a thick building of type D_n we set $\tilde{X} = \tilde{X}(V) = \{U \in X \mid \dim U \neq n - 1\}$ and we say that $U, W \in \tilde{X}$ are incident if and only if

$$U \subseteq W \text{ or } W \subseteq U \text{ or } \dim(W \cap U) = n - 1.$$

Then $\tilde{\Delta} = \mathrm{Flag}\, \tilde{X}$ is the desired building. We will write $\mathrm{Orifl}(\tilde{X})$ for $\mathrm{Flag}\, \tilde{X}$ to stress that we consider $\tilde{X}$ not with the usual incidence relation given by inclusion, but with this new one. Note that $\mathrm{Orifl}\, \tilde{X}$ is called the oriflamme complex of $\tilde{X}$.

For $n \geq 4$ every building of type D_n is of the form $\tilde{\Delta}$ for some field K [Tit74, Proposition 8.4.3]. We also consider the construction for $n = 2, 3$ in which case we get certain (but not all) buildings of type $D_2 = A_1 \times A_1$ and $D_3 = A_3$, which where already covered earlier.

To describe the opposite complexes, we need to introduce some further notation.

Definition 5.9. For arbitrary subspaces U, W of V, we define

$$U \tilde{\pitchfork}_V W \iff U \pitchfork_V W \text{ or } (U, W \in \tilde{X}, U = U^\perp, W = W^\perp \text{ and } \dim(U \cap W) = 1).$$

For a set $\mathcal{E}$ of subsets of V we define

$$X_{\mathcal{E}}(V) = \{U \in X \mid U \tilde{\pitchfork} \mathcal{E}\}, \qquad T_{\mathcal{E}}(V) = \mathrm{Flag}\, X_{\mathcal{E}}(V)$$
$$\tilde{X}_{\mathcal{E}}(V) = \{U \in \tilde{X} \mid U \tilde{\pitchfork} \mathcal{E}\}, \qquad \tilde{T}_{\mathcal{E}}(V) = \mathrm{Orifl}\, \tilde{X}_{\mathcal{E}}(V).$$

5.2.2 Cone functions for a subdivision

With the above defined notation, we can now describe how to construct a 1-cone function for $\tilde{T}_{\mathcal{E}}(V)$ given a 1-cone function for $T_{\mathcal{E}}(V)$.

Proposition 5.10. *Let* $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ *and set* $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$. *If* $T_{\mathcal{E}(\tau)}(V)$ *has an 1-cone function* $\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}$ *with* $\mathrm{Rad}_i(\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}) \leq c, i = 0, 1$ *for some* $c \geq 0$ *then there exists a 1-cone function* $\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}$ *with* $\mathrm{Rad}_0(\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}) \leq c$ *and* $\mathrm{Rad}_1(\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}) \leq 2c$.

To prove the proposition, we need the following definition and auxiliary lemmas.

Definition 5.11. Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$. Denote $T = T_{\mathcal{E}(\tau)}(V)$ and $\tilde{T} = \tilde{T}_{\mathcal{E}(\tau)}(V)$. Each $u \in X$ with $\dim u = n - 1$ is contained in two n-dimensional spaces in X. We denote them by W_u and $\bar{W}_u$.

We define the following function:

$$\varphi : T_{\mathrm{ord}}(1) \to \text{ paths in } \tilde{T}$$

$$(u, v) \mapsto \begin{cases} (W_u; v) & \text{if } \dim u = n - 1, v \neq W_u \\ (W_u) & \text{if } \dim u = n - 1, v = W_u \\ (u; W_v) & \text{if } \dim v = n - 1, u \neq W_v \\ (W_v) & \text{if } \dim v = n - 1, u = W_v \\ (u; v) & \text{else.} \end{cases}$$

Note that this definition is well defined, since if (u, v) is an edge in T and $\dim v = n - 1$ then (u, W_v) is an edge in $\tilde{T}$ (either $u \subseteq v \subseteq W_v$ or $u \cap W_v = v$).

Lemma 5.12. *The map φ preserves paths, i.e. given a path $P = (v_0; \dots; v_m)$ in T, we have that $\varphi(P) := \varphi(v_0, v_1) \circ \varphi(v_1, v_2) \circ \dots \circ \varphi(v_{m-1}, v_m)$ is a path in $\tilde{T}$.*

Proof. Given two edges $(u, v), (v, w) \in T_{\mathrm{ord}}(1)$ such that the terminal vertex of one edge equals the initial vertex of the other. Then the terminal vertex of $\varphi((u, v))$ will be equal to the initial vertex of $\varphi((v, w))$ (if $\dim v \neq n - 1$ it will be v, otherwise it will be W_v). Hence, the concatenation of the images of the edges of a path is well defined and gives again a path. $\square$

Note that it follows from the definition that $\varphi(P \circ Q) = \varphi(P) \circ \varphi(Q)$ and $\varphi(P^{-1}) = \varphi(P)^{-1}$.

Lemma 5.13. *Given two paths P, P' in T, if $P \sim_{(BT)} P'$ then $\varphi(P) \sim_{(BT)} \varphi(P')$ or $\varphi(P) = \varphi(P')$.*

Proof. Since $P \sim_{(BT)} P'$ there exit paths Q_0, Q_1 and vertices u, v in T such that $P = Q_0 \circ (u; v; u) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u) \circ Q_1^{-1}$. Thus, $\varphi(P) = \varphi(Q_0) \circ \varphi((u; v; u)) \circ \varphi(Q_1)^{-1}$ and similarly for $\varphi(P')$. Now, if $\dim u, \dim v \neq n - 1$ then $\varphi((u; v; u)) = (u; v; u)$ and $\varphi((u)) = (u)$ which proves the statement in this case.

If $\dim u = n - 1$ then $\varphi((u)) = (W_u)$ and $\varphi((u; v; u)) = (W_u; v; W_u)$ or $\varphi((u; v; u)) = (W_u)$ and the statement follows.

If $\dim v = n - 1$ then $\varphi((u)) = (u)$ and $\varphi((u; v; u)) = (u; W_v; u)$ or $\varphi((u; v; u)) = (W_v) = (u)$ (the second case occurs exactly when $u = W_v$). $\square$

Lemma 5.14. *Given two paths P, P' in T, if $P \sim_{(TR)} P'$ then $\varphi(P) \sim_{(TR)} \varphi(P')$, $\varphi(P) \sim_{(BT)} \varphi(P')$ or $\varphi(P) = \varphi(P')$.*

Proof. Since $P \sim_{(TR)} P'$ there exit paths Q_0, Q_1 and a triangle (u, v, w) in T such that $P = Q_0 \circ (u; v; w) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u; w) \circ Q_1^{-1}$. Thus, $\varphi(P) = \varphi(Q_0) \circ \varphi((u; v; w)) \circ \varphi(Q_1)^{-1}$ and similarly for $\varphi(P')$.

We will proceed by case distinction which we display in the following table. The last column shows the relationship between $\varphi(P)$ and $\varphi(P')$.

u	v	w	$\varphi((u;v;w))$	$\varphi((u;w))$	$\varphi(P)$ vs $\varphi(P')$
$\dim u \neq n-1$	$\dim v \neq n-1$	$\dim w \neq n-1$	$(u;v;w)$	$(u;w)$	$\sim_{\mathrm{TR}}$
$u \neq W_w$	$v \neq W_w$	$\dim w = n-1$	$(u;v;W_w)$	$(u;W_w)$	$\sim_{\mathrm{TR}}$
$u \neq W_w$	$v = W_w$	$\dim w = n-1$	$(u;W_w)$	$(u;W_w)$	$=$
$u = W_w$	$v \neq W_w$	$\dim w = n-1$	$(u;v;W_w)$	(W_w)	$\sim_{\mathrm{BT}}$
$u \neq W_v$	$\dim v = n-1$	$w \neq W_v$	$(u;W_v;w)$	$(u;w)$	$\sim_{\mathrm{TR}}$
$u \neq W_v$	$\dim v = n-1$	$w = W_v$	$(u;W_v)$	$(u;w = W_v)$	$=$
$u = W_v$	$\dim v = n-1$	$w \neq W_v$	$(W_v;w)$	$(u = W_v;w)$	$=$
$\dim u = n-1$	$v \neq W_u$	$w \neq W_u$	$(W_u;v;w)$	$(W_u;w)$	$\sim_{\mathrm{TR}}$
$\dim u = n-1$	$v = W_u$	$w \neq W_u$	$(W_u;w)$	$(W_u;w)$	$=$
$\dim u = n-1$	$v \neq W_u$	$w = W_u$	$(W_u;v;w)$	(W_u)	$\sim_{\mathrm{BT}}$

$\square$

Proof of Proposition 5.10. We consider the setting of Definition 5.11. Assume that we have a 1-cone function $(v_0, (P_v)_{v \in T(0)}, (T_{(u,v)})_{(u,v) \in T_{\mathrm{ord}}}(1))$ of T. The goal is to construct a 1-cone function $(\tilde{v}_0, (\tilde{P}_v)_{v \in \tilde{T}(0)}, (\tilde{T}_{(u,v)})_{(u,v) \in \tilde{T}_{\mathrm{ord}}(1)})$ for $\tilde{T}$.

For the apex, we set $\tilde{v}_0 = v_0$ if $\dim v_0 \neq n-1$ and otherwise $\tilde{v}_0 = W_{v_0}$.

Given a vertex $v \in \tilde{T}(0) = \tilde{Y}$, we set $\tilde{P}_v = \varphi(P_v)$. Since $v \in \tilde{Y}$, we have $\dim v \neq n-1$ and hence $\tilde{P}_v$ is a path from $\tilde{v}_0$ to v of length at most the length of P_v.

Let $(u,v) \in \tilde{T}_{\mathrm{ord}}(1)$. To define the contraction $\tilde{T}_{(u,v)}$, we distinguish two cases. First assume that $(u,v) \in T \cap \tilde{T}$, i.e. at least one of the two vertices has dimension less than n. Hence we can consider $T_{(u,v)} = (P_0, \ldots, P_m)$ and define $\tilde{T}_{(u,v)} = (\varphi(P_0), \ldots, \varphi(P_m))$. The auxiliary Lemmas 5.12, 5.13, 5.14 imply that this is a well defined contraction and $|\tilde{T}_{(u,v)}| \leq |T_{(u,v)}|$.

For the second case, let $(u,v) \in \tilde{T}_{\mathrm{ord}}(1)$ with $\dim u = \dim v = n$. Consider $T_{(u,u \cap v)} = (P_0^1, \ldots, P_m^1)$ and $T_{(u \cap v, v)} = (P_0^2, \ldots, P_\ell^2)$. Note that $\varphi((u; u \cap v; v)) = (u,v)$. We set

$$\tilde{P}_0 = \tilde{P}_u \circ \varphi((u, u \cap v) \circ (u \cap v, v)) \circ \tilde{P}_v^{-1} = \varphi(P_u \circ (u; u \cap v; v) \circ P_v^{-1}).$$

By $|P_{u \cap v}| =: s$ many backtracking steps, we can transform $\tilde{P}_0$ to

$$\tilde{P}_s = \varphi(P_u \circ (u, u \cap v) \circ P_{u \cap v}^{-1} \circ P_{u \cap v} \circ (u \cap v, v) \circ P_v^{-1}) = \varphi(P_0^1 \circ P_0^2).$$

Applying the contraction steps from $T_{(u, u \cap v)}$ on the first half of the path and then the steps from $T_{(u \cap v, v)}$ on the second half we get the desired contraction, (again using Lemma 5.13 and Lemma 5.14). In particular $|\tilde{T}_{(u,v)}| \leq |T_{(u, u \cap v)}| + |T_{(u \cap v, v)}|$. This also proves the bound on the 1-cone radius.

$\square$

Remark 5.15. The above reasoning also shows that if T has a 0-cone function, then we can construct a 0-cone function for $\tilde{T}$. Further note that the dimensions of T and $\tilde{T}$ coincide. Hence, we can also get a more general statement of Proposition 5.10 saying that given a NAC for T, we can construct a NAC for $\tilde{T}$ and $\mathrm{Rad}(\tilde{T}) \leq 2\,\mathrm{Rad}(T)$.

5.2.3 Induction argument for the type D_n case

For the D_n case, we need the following weaker and more specialized version of Theorem 5.5. In order to state it, we need the following definition. See [OVB25, Theorem 6.13] for the Abelian version.

Definition 5.16. Let K be a field, V a K-vector space, Y a collection of subspaces of V. Let $X = \operatorname{Flag} Y$ and $U \in Y$. Then we define

$$Y^{<U} := \{W \in Y \mid W < U\}, \ X^{<U} := \operatorname{Flag} Y^{<U}$$
$$Y^{>U} := \{W \in Y \mid W > U\}, \ X^{>U} := \operatorname{Flag} Y^{>U}.$$

Remark 5.17. Note that

$$\operatorname{lk}_X(U) = X^{<U} * X^{>U}.$$

Theorem 5.18. *Let $\mathbf{C}$ be a class of tuples $(\kappa, (V, Q, f))$ where (V, Q, f) is a thick pseudo-quadratic space and $\kappa = \operatorname{Flag} Y$ where $Y \neq \emptyset$ is a set of vector subspaces of V. Assume that for any $n \in \mathbb{N}, n \geq 2$ there exist $\ell_n \in \mathbb{N}$ such that for all $\kappa \in \mathbf{C}, \dim \kappa = n$ there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa, \ell \leq \ell_n$ satisfying:*

(i) There exists an 1-cone function $\operatorname{Cone}_{\kappa_0}$ such that for $j = 0, 1$

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq f(n),$$

where $f : \mathbb{N} \to \mathbb{R}$ is a function depending only on $\mathbf{C}$.

(ii) Every κ_i is a full subcomplex of κ, i.e. $\kappa_i = \operatorname{Flag} Y_i$ for some $Y_i \subseteq Y$ and all vector subspaces in $Y_i \setminus Y_{i-1}$ have the same dimension.

(iii) For every $U \in Y_i \setminus Y_{i-1}$ we have that $\operatorname{lk}_{\kappa_i}(U) \cap \kappa_{i-1} = \kappa_{i-1}^{<U} * \kappa_{i-1}^{>U}$ is such that $\kappa_{i-1}^{<U}, \kappa_{i-1}^{>U} \in \mathbf{C}$ or one (or both) have again a filtration $R_0 \subset \ldots R_h, h \leq \ell_{i-1}$ which starts with an element of $\mathbf{C}$ (i.e. $R_0 \in \mathbf{C}$), such that Condition 2. is satisfied and such that for every $W \in R_j(0) \setminus R_{j-1}(0)$ we have $R_{j-1}^{<W}, R_{j-1}^{>W} \in \mathbf{C}$.

Furthermore, we assume that there exists $\ell_1 \in \mathbb{N}$ such that for any $\kappa \in \mathbf{C}$ with $\dim(\kappa) = 1$, there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ of κ of length $\ell \leq \ell_1$ such that the following is satisfied (set $V_i := \{\text{vertices of } \kappa_i\} \setminus \{\text{vertices of } \kappa_{i-1}\}$):

(i) There is an 0-cone function $\operatorname{Cone}_{\kappa_0}$ such that

$$\operatorname{Rad}_0(\operatorname{Cone}_{\kappa_0}) \leq f(1),$$

(ii) κ_i is a full subcomplex of κ and for all vertices $w_1, w_2 \in V_i$ we have $\{w_1, w_2\} \notin \kappa$, for all $0 \leq i \leq \ell$.

(iii) $\operatorname{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ is non-empty for all $w \in V_i, 1 \leq i \leq \ell$.

Then for every $n \in \mathbb{N}, n \geq 2$ there exists a constant $\mathcal{R}(n)$ such that for every $\kappa \in \mathbf{C}$ with $\dim \kappa = n$ we have: κ has a 1-cone function $\operatorname{Cone}_\kappa$ such that for $j = 0, 1$ we have

$$\operatorname{Rad}_j(\operatorname{Cone}_\kappa) \leq \mathcal{R}(n).$$

Furthermore, every $\kappa \in \mathbf{C}$ with $\dim(\kappa) = 1$ admits a 0-cone function with $\operatorname{Rad}_0(\operatorname{Cone}_\kappa) \leq f(1) + \ell_1$.

Proof. The proof is analogues to the one of Theorem 5.5. The main difference is that to get a cone function for $\mathrm{lk}_{\kappa_i}(U) \cap \kappa_{i-1}$ it might happen that we cannot just use the induction hypothesis, but that we need to do another induction on the filtration of $\kappa_{i-1}^{<U}$ or $\kappa_{i-1}^{>U}$, using Theorem 4.13 in the step of the induction. This gives an even weaker and more complicated bound on the cone radius of the final cone function, thus we decided to not keep track of it. But it will still only depend on n and the class $\mathbf{C}$, not on the field K. $\qquad\square$

Remark 5.19. Similar to Theorem 5.5, if we are give a class of simplicial complexes $\mathbf{C}$ which is know to satisfy the assumptions of the Abelian version of this theorem, [OVB25, Theorem 6.13], then it suffices to check that the κ_0 appearing in the filtrations of the Abelian version also admit a non-Abelian 1-cone function (resp. 0-cone function if κ_0 is one-dimensional).

We get the following corollary, details about the terminology and the proof can be found in Appendix D.

Corollary 5.20 (Corollary D.12). *For every $n \in \mathbb{N}, n \geq 2$ there exist a constant $\mathcal{R}(n-1)$ such for every building $\tilde{\Delta}$ of type D_n over a field K with $|K| \geq 2^{2n-1}$ and $a \in \tilde{\Delta}$ there exists a NAC $\mathrm{Cone}_{\tilde{\Delta}^0(a)}$ of $\tilde{\Delta}^0(a)$ such that*
$$\mathrm{Rad}(\mathrm{Cone}_{\tilde{\Delta}^0(a)}) \leq 2\mathcal{R}(n-1).$$

5.3 Proof of Theorem 5.1

The above bounds on the cone radii of NAC of opposite complexes yields the proof of Theorem 5.1:

Proof. Let $n \geq 3$ be an integer and $q > 2^{2n-1}$ be a prime power. For every n-dimensional, n-classical KMS complex X over $\mathbb{F}_q$ every link of a vertex of X is either an opposite complex of a classical type, i.e., of type A_n, B_n, C_n or D_n or a join of two such opposite complexes. By Corollaries B.4, C.2, D.12 it follows that the link is a strongly symmetry simplicial complex that has a non-Abelian cone function whose radii are bounded independently of q (in the case of the join, each factor has a non-Abelian cone function whose radii are bounded independently of q and thus by Proposition 4.9, the join also has such cone function). Thus by Lemma 4.4 it follows that every link is a 1-coboundary expander with respect to any group Λ and that there is a bound on the expansion that is independent of q. $\qquad\square$

A Buildings, their opposite complexes and relation to KMS complexes

This section is almost identical with [OVB25, Chapter 5]. We include it to improve the self-containedness of this paper and to recall the necessary notations.

A.1 Buildings and BN-pairs

One way to define a building is as a simplicial complex with a family of subcomplexes called apartments. Each apartment is isomorphic to the same Coxeter complex. The type of the Coxeter complex, which can be described by a root system/ a GCM, is also what we call the type of the building. Each two simplices of the building have to be contained in a common apartment and each two apartments are isomorphic with an isomorphism that acts as the identity on the intersection.

One way to construct buildings is via groups with a BN-pair (and for thick, irreducible buildings of dimension 2 and larger, all buildings can be constructed that way, as was proven by Jacques Tits in [Tit74]).

Given a group G, a BN-pair of G is a pair of subgroups $B, N \leq G$ satisfying certain properties, see [AB08, Definition 6.55]. In particular, $G = BN$ and setting $T = B \cap N$ we have that $W = N/T$ admits a set of generators S, such that (W, S) is a Coxeter system. W is also called the Weyl group of the BN-pair. To describe the building, we need to define the parabolic subgroups $P_J := B\langle J \rangle B$ for $J \subset S$ (here $\langle J \rangle$ is the subgroup generated by the elements of J inside W). The building can then we described as the following coset complex:

$$\Delta(G, B) := \mathcal{CC}(G; (P_{S \setminus \{j\}})_{j \in S}).$$

Since $P_J \cap P_L = P_{J \cap L}$ for any $J, L \subseteq S$ we can see $\Delta(G, B)$ also as the poset $\{gP_J \mid g \in G, J \subseteq S\}$ ordered by reversed inclusion. Then the simplex $\{gP_{S \setminus \{j\}} \mid j \in J\} \in \mathcal{CC}(G; (P_{S \setminus \{j\}})_{j \in S})$ corresponds to $gP_{S \setminus J}$ for $J \subseteq S, g \in G$. In particular, the set of maximal simplices, called chambers, corresponds to G/B.

If the Weyl group is finite, we call the building spherical. This coincides with our definition of calling a GCM/root system spherical. A finite Weyl group has a unique longest element denoted by w_0, where longest means that is generated by the largest number of generators, counted with multiplicities.

Two chambers $gB, hB, g, h \in G$ are called opposite if and only if $g^{-1}h \in Bw_0B$, denoted by gB op hB.

Given a simplex $\sigma \in \Delta(G, B)$, the complex opposite σ is defined as follows:

$$\Delta^0(\sigma) := \{\tau \in \Delta(G, B) \mid \text{ there exist chambers } c, d \in \Delta(G, B) \text{ such that } \tau \subseteq c, \sigma \subseteq d, c \text{ op } d\}.$$

To see the connection between opposite complexes and the links of KMS complexes, note that every Chevalley group has a BN-pair in the following way. Let $A = (A_{ij})_{i,j \in I}$ be a spherical GCM of rank at least 2, K a field and $G = \text{Chev}_A(K) = \langle x_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K \rangle$, using the notation of Definition 2.6. We set

$$U_\alpha := \langle x_\alpha(\lambda) \mid \lambda \in K \rangle, \alpha \in \Phi, \qquad U^+ := \langle U_\alpha \mid \alpha \in \Phi^+ \rangle, \qquad T := \langle h_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K \rangle,$$
$$B := U^+T, \qquad\qquad N := \langle T, n_{-\alpha_i}(-\lambda^{-1}); i \in I, \lambda \in K^* \rangle.$$

For example [AB08, Theorem 7.115] shows that this is indeed a BN-pair.

Let $C_0 = 1B$ denote the fundamental chamber of $\Delta(G, B)$ where G is a Chevalley group as described above. Recall that w_0 denotes the longest element on the Weyl group W. It satisfies $Bw_0B = U^+w_0B$. We can describe the opposite complex in this set-up as

$$\Delta^0(C_0) = \{gP_J \mid g \in G, J \subseteq S : \exists h \in U^+w_0B : gP_J \supseteq hB\}$$
$$= \{gP_J \mid \exists h \in U^+w_0B : h \in gP_J\}$$
$$= \{hP_J \mid h \in U^+w_0B\}$$
$$= \{uw_0P_J \mid u \in U^+\}.$$

This leads to the following proposition.

Proposition A.1. *[GdPVB25, Chapter 3.2] Let G be a Chevalley group with BN-pair as described above. Then*

$$\mathcal{CC}(U^+, (U_{I \setminus \{i\}})_{i \in I}) \cong \Delta^0(C_0).$$

If we assume that A is 2-spherical and $|K| \geq 4$ we have $U^+ = \langle U_{\alpha_i} \mid i \in I \rangle$ (see e.g. the comment before [GdPVB25, Lemma 3.7]). Combining this with the above proposition and Fact 2.14 gives the following corollary.

Corollary A.2. *Let X be a KMS complex, and $\sigma \in X$ a face of dimension less than or equal to $\dim X - 2$ of type $\emptyset \neq T \subset I$. Then*
$$\mathrm{lk}_X(\sigma) \cong \Delta^0(C_0)$$
where Δ is the building obtained from the BN-pair structure of the Chevalley group of type $A_{I \setminus T}$.

A.2 Geometric constructions of buildings

In the following section, we describe how buildings of type A_n, C_n and D_n can be constructed as flag complexes of certain sets equipped with an incidence relation. We furthermore describe the opposite complexes in this set-up. The general framework is as follows.

Definition A.3. Let X be a set and let $I \subseteq X \times X$ be a reflexive and symmetric relation. For $a, b \in X$ we write aIb if and only if $(a, b) \in I$ and call a and b incident. The structure gives rise to the following simplicial complex, called the flag complex of X:

$$\mathrm{Flag}(X) = \{\sigma \subseteq X \mid \forall a, b \in \sigma : aIb\}.$$

A.2.1 Buildings of type A_n

Let $n \in \mathbb{N}, n \geq 1$. Any building of type A_n can be described in the following way, see e.g. [Tit74]. Let K be a field or skew field, V an $(n+2)$-dimensional vector space over K. Set $X := X(V) := \{0 < U < V \mid U \text{ is a } K\text{-subspace of } V\}$. Two subspaces $U, W \in X$ are called incident if $U \subseteq W$ or $W \subseteq U$. Then $\Delta = \mathrm{Flag}\, X$ is a building of type A_n.

To describe the opposite complex in this context, we need the following notation, see [Abr96, Definition 9].

Definition A.4. (i) Two subspaces U, W of V are called transversal (in V) if $U \cap W = 0$ or $U + W = V$. If this is the case, we write $U \pitchfork_V W$ or simply $U \pitchfork W$.

(ii) Let $\mathcal{E}$ be a set of subspaces of V. Then we write $U \pitchfork \mathcal{E}$ if $U \pitchfork E$ for all $E \in \mathcal{E}$. We set

$$X_{\mathcal{E}}(V) := \{U \in X \mid U \pitchfork \mathcal{E}\} \qquad T_{\mathcal{E}}(V) = \mathrm{Flag}\, X_{\mathcal{E}}(V).$$

We get the following result.

Proposition A.5. *[Abr96, Corollary 12] For any simplex $\sigma = \{E_1 < \cdots < E_r\} \in \Delta$ set $\mathcal{E}(\sigma) = \{E_i \mid 1 \leq i \leq r\}$. Then*
$$\Delta^0(\sigma) = T_{\mathcal{E}(\sigma)}(V).$$

Definition A.6. We denote by $\mathbf{C}_A$ the class of all simplicial complexes $T_{\mathcal{E}}(V)$ where $\dim(T_{\mathcal{E}}(V)) = n \geq 0$, K is a (skew) field, V an $(n+2)$-dimensional vector space over K, $\mathcal{E}$ is a finite set of subspaces of V such that if we set $e_j = |\{E \in \mathcal{E} \mid \dim E = j\}|$ we have $\sum_{j=1}^{n+1} \binom{n}{j-1} e_j \leq |K|$. Furthermore, $T_{\mathcal{E}}(V)$ is defined as described in Definition A.4.

A.2.2 Hermitian and pseudo-quadratic forms

In this part, we recall facts about hermitian and pseudo-quadratic forms that are needed to describe the buildings of type C_n and D_n. We follow [Abr96, Chapter 5], who in turn mostly follows [Tit74].

Let K be a skew field, $\sigma : K \to K, a \mapsto a^\sigma$ be an involution, i.e. an anti-automorphism of K such that $\sigma^2 = \mathrm{id}_K$. Let $\varepsilon \in \{1, -1\} \subset K$. If $\sigma \neq \mathrm{id}_K$ then we require $\varepsilon = -1$. Let V be a right K-vector space of dimension $m \in \mathbb{N} \cup \{\infty\}$. Let $K_{\sigma,\varepsilon} := \{\alpha - \alpha^\sigma \varepsilon \mid \alpha \in K\}$ and $K^{\sigma,\varepsilon} := \{\alpha \in K \mid \alpha + \alpha^\sigma \varepsilon = 0\}$. Let Λ be a form parameter relative to (σ, ε), i.e. Λ is a subgroup of $(K, +)$ satisfying $K_{\sigma,\varepsilon} \subseteq \Lambda \subseteq K^{\sigma,\varepsilon}$ and $\alpha^\sigma \Lambda \alpha \subseteq \Lambda$ for all $\alpha \in K$.

Let $f : V \times V \to K$ be a (σ, ε)-hermitian form, i.e. f is biadditive and for all $x, y \in V, a, b \in K$ we have $f(xa, yb) = a^\sigma f(x, y) b$ and $f(y, x) = f(x, y)^\sigma \varepsilon$. Let $Q : V \to K/\Lambda$ be a (σ, ε)-quadratic form with associated (σ, ε)-hermitian form f, i.e. $Q(xa) = a^\sigma Q(x) a + \Lambda$ and $Q(x + y) - Q(x) - Q(y) = f(x, y) + \Lambda$ for all $x, y \in V, a \in K$. If $\Lambda = K$ we require that f is alternating. If $\Lambda \neq K$ then f is uniquely defined by Q.

For $M \subseteq V$ we write $M^\perp := \{x \in V \mid f(x, M) = 0\}$. A subspace $U < V$ is called

- non-degenerate if $U \cap U^\perp = 0$,

- totally degenerate if $U \subseteq U^\perp$,

- anisotropic if $0 \notin Q(U \setminus \{0\})$,

- isotropic if $0 \in Q(U \setminus \{0\})$,

- totally isotropic if $U \subseteq U^\perp$ and $Q(U) = 0$.

We require that there is at least one finite-dimensional maximal totally isotropic subspace. In that case, all maximal totally isotropic subspace have the same dimension. We denote this dimension by n and call it the Witt index of (V, Q, f). We require $0 < n < \infty$. Furthermore, we require that $V^\perp = 0$. We say that the triple (V, Q, f) is a pseudo-quadratic space if it is of the form described above.

If it additionally satisfies $(m, \Lambda) \neq (2n, 0)$ then we say (V, Q, f) is a thick pseudo-quadratic space.

Note that if (V, Q, f) is a thick pseudo-quadratic space and K is a finite field, then $\dim V \leq 2n + 1$. In general, if one only considers finite fields, especially of characteristic different from 2, the set-up can be substantially simplified, see [AB08, Remark 9.3, 9.4].

A.2.3 Buildings of type C_n

Let $n \in \mathbb{N}, n \geq 1$ and (V, Q, f) be a thick pseudo-quadratic space with Witt index n. Set $X := X(V) = \{0 < U < V \mid U \text{ is totally isotropic}\}$ with incidence relation given by containment as in the A_n case and $\Delta = \mathrm{Flag}\, X$. Then Δ is a thick building of type C_n and every classical C_n building can be obtained in this way (here classical means that the links of type A_2 correspond to Desarguesian planes, which is always the case for $n \geq 4$), see [Tit74, Theorem 8.22].

Definition A.7. Set $\mathcal{U} := \{0 \leq U \leq V \mid \dim U < \infty\}, \mathcal{U}^\perp := \{U^\perp \mid u \in \mathcal{U}\}$ and $\mathcal{W} := \mathcal{U} \cup \mathcal{U}^\perp$.

Let $\mathcal{E}$ be a finite subset of $\mathcal{W}$ such that $\mathcal{E}^\perp = \mathcal{E}$. Assume $K = \mathbb{F}_q$ is a finite field. Set $\mathcal{E}_j := \{E \in \mathcal{E} \mid \dim E = j\}, e_j := |\mathcal{E}_j|$ and $e_h^{(s)} := \sum_{j=0}^{2s} \binom{2s}{j} e_{h+j}$ for $h \in \mathbb{N}, s \in \mathbb{N}_0$ and $h + 2s < m$. We define

- $N(\mathcal{E}) := e_1^{(n-1)}$ if f is alternating ($\Rightarrow m = 2n$) and $Q = 0$,

- $N(\mathcal{E}) := \left(e_1^{(n-1)}\right)^2$ if $m = 2n$ and $\sigma \neq \mathrm{id}$,

- $N(\mathcal{E}) := 2e_2^{(n-1)}$ if $m = 2n + 1$,

- $N(\mathcal{E}) := \max\left\{e_2^{(n-1)} + e_3^{(n-1)} + 1, 2e_3^{(n-1)}\right\}$ if $m = 2n + 2$.

Definition A.8. For a set $\mathcal{E}$ of subspaces of V we define, similar to Definition A.4,

$$X_{\mathcal{E}}(V) := \{U \in X \mid U \pitchfork \mathcal{E}\}, \qquad T_{\mathcal{E}}(V) = \mathrm{Flag}\, X_{\mathcal{E}}(V).$$

Proposition A.9. *[Abr96, Corollary 15] For any simplex* $\tau = \{E_1 < \cdots < E_r\} \in \Delta$ *set* $\mathcal{E}(\tau) = \{E_i, E_i^{\perp} \mid 1 \leq i \leq r\}$. *Then* $\Delta^0(\tau) = T_{\mathcal{E}(\tau)}(V)$.

Definition A.10. With $\mathbf{C}_C$ we denote the class of simplicial complexes $T_{\mathcal{E}}(V)$ where $n = \dim(T_{\mathcal{E}}(V) + 1 \geq 1$, K a (skew) field, (V, Q, f) is a thick pseudo-quadratic space with Witt index n, $\mathcal{E}$ a finite subset of $\mathcal{W}$ (see Definition A.7) such that $\mathcal{E}^{\perp} = \mathcal{E}$. If K is a finite field, we further require that $|K| \geq N(\mathcal{E})$. We set $T_{\mathcal{E}}(V)$ as in Definition A.8.

A.2.4 Buildings of type D_n

For this section let (V, Q, f) be a pseudo-quadratic space but in the specific case where K is a field, $\sigma = \mathrm{id}_K, \varepsilon = 1, \Lambda = 0$, V a K-vector space of dimension $m = 2n \geq 4$, Q an ordinary quadratic form and f the corresponding symmetric bilinear form. We further assume that V is non-degenerate and of Witt index n.

We set as before $X = X(V) = \{0 < U < V \mid U$ is totally isotropic$\}$. Then $\Delta = \mathrm{Flag}\, X$ is a weak C_n building, in particular it is not thick, since each totally isotropic subspace of dimension $n - 1$ contained in exactly two totally isotropic subspaces of dimension n. Furthermore, the space $Y = \{U \in X \mid \dim U = n\}$ is partitioned into to sets $Y = Y_1 \sqcup Y_2$ such that for $U_1, U_2 \in Y_i$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z}$ for $i = 1, 2$ and for $U_1 \in Y_1, U_2 \in Y_2$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z} + 1$. In particular, if for $U_1, U_2 \in Y$ we have $\dim(U_1 \cap U_2) = n - 1$ then U_1 and U_2 are in different sets of the partition.

To obtain a thick building of type D_n we set $\tilde{X} = \tilde{X}(V) = \{U \in X \mid \dim U \neq n - 1\}$ and we say that $U, W \in \tilde{X}$ are incident (write UIW) if and only if

$$U \subseteq W \text{ or } W \subseteq U \text{ or } \dim(W \cap U) = n - 1.$$

Then $\tilde{\Delta} = \mathrm{Flag}\, \tilde{X}$ is desired building. We will write $\mathrm{Orifl}(\tilde{X})$ for $\mathrm{Flag}\, \tilde{X}$ to stress that we consider $\tilde{X}$ not with the usual incidence relation given by inclusion, but with this new one. Note that $\mathrm{Orifl}\, \tilde{X}$ is called the oriflamme complex of $\tilde{X}$.

For $n \geq 4$ every building of type D_n is of the form $\tilde{\Delta}$ for some field K [Tit74, Proposition 8.4.3]. We also consider the construction for $n = 2, 3$ in which case we get certain (but not all) buildings of type $D_2 = A_1 \times A_1$ and $D_3 = A_3$, which where already covered earlier.

To describe the opposite complexes, we need to introduce some further notation.

Definition A.11. For arbitrary subspaces U, W of V, we define

$$U \tilde{\pitchfork}_V W \iff U \pitchfork_V W \text{ or } (U, W \in \tilde{X}, U = U^\perp, W = W^\perp \text{ and } \dim(U \cap W) = 1).$$

For a set $\mathcal{E}$ of subsets of V we define (different from Definition A.4)

$$X_{\mathcal{E}}(V) = \{U \in X \mid U \tilde{\pitchfork} \mathcal{E}\}, \qquad T_{\mathcal{E}}(V) = \text{Flag}\, X_{\mathcal{E}}(V)$$
$$\tilde{X}_{\mathcal{E}}(V) = \{U \in \tilde{X} \mid U \tilde{\pitchfork} \mathcal{E}\}, \qquad \tilde{T}_{\mathcal{E}}(V) = \text{Orifl}\, \tilde{X}_{\mathcal{E}}(V)$$

Proposition A.12. *[Abr96, Corollary 17] Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \le i \le r\}$ then $\tilde{\Delta}^0(\tau) = \tilde{T}_{\mathcal{E}(\tau)}(V)$.*

B   Details for type A_n

In this section, we fill in the details of the proof that the complex opposite the fundamental chamber in a spherical building of type A_n has a non-Abelian cone function with bounded cone radius, by showing that the class $\mathbf{C}_A$ from Definition A.6 satisfies the assumptions of Theorem 5.5. The argument will follow [OVB25, Chapter 6.2] closely.

We will need some facts that appear in the proof of [Abr96, Proposition 12].

Fact B.1. If $T_{\mathcal{E}}(V) \in \mathbf{C}_A$, then there exists $\ell \in X_{\mathcal{E}}(V)$ such that ℓ is a line, i.e., $\dim(\ell) = 1$.

Let $\ell \in X_{\mathcal{E}}(V)$ be a line. Define

$$Y_0 = \{U \in T_{\mathcal{E}}(V) : \ell + U \in T_{\mathcal{E}}(V)\},$$

and further define κ_0 to be the subcomplex of $T_{\mathcal{E}}(V)$ spanned by Y_0.

Assume now that $\dim V = n + 2$ and hence $\dim T_{\mathcal{E}}(V) = n$ (and $\text{Flag}\, X$ is a building of type A_{n+1}). For $1 \le i \le n + 1$, define $Y_i \subseteq X_{\mathcal{E}}(V)$ as follows:

$$Y_i = \{U \in X_{\mathcal{E}}(V) \mid \dim(U) \ge n + 2 - i \text{ or } U \in Y_0\}.$$

For every such i, let κ_i be the subcomplex of $T_{\mathcal{E}}(V)$ spanned by Y_i.

Fact B.2. We denote $Y = X_{\mathcal{E}}(V), \kappa = T_{\mathcal{E}}(V)$ and for every $U \in T_{\mathcal{E}}(V)$ we denote the link of U by $\text{lk}_\kappa(U)$. For every $1 \le i \le n + 1$ and every $U \in Y_i \setminus Y_{i-1}$, it holds that $\text{lk}_\kappa(U) \cap \kappa_{i-1} = T_{\mathcal{E}'}(U) * T_{\overline{\mathcal{E}}}(V/U)$ such that

- The set $\mathcal{E}'$ is a set of subsets of U satisfying $\sum_{j=1}^{n+1} \binom{n}{j-1} e_j' \le |K|$, hence $T_{\mathcal{E}'}(U) \in \mathbf{C}_A$.

- The set $\overline{\mathcal{E}}$ is a set of subsets of V/U that satisfying $\sum_{j=1}^{n+1} \binom{n}{j-1} \overline{e}_j \le |K|$ and hence $T_{\mathcal{E}'}(V/U) \in \mathbf{C}_A$.

Lemma B.3. *There is a NAC Cone_{κ_0} such that*

$$\text{Rad}(\text{Cone}_{\kappa_0}) \le n + 2.$$

Proof. Let Γ_0 be the subcomplex of κ_0 spanned by $Z_0 = \{U \in X_{\mathcal{E}}(V) : \ell \leq U\}$. For $1 \leq i \leq n+1$, we denote

$$Z_i = \{U \in Y_0 \mid \dim(U) \leq i \text{ or } U \in Z_0\}.$$

Further denote $\Gamma_i = \operatorname{Flag} Z_i$. Note that $\Gamma_{n+1} = \kappa_0$. We will show by induction that for every $0 \leq i \leq n+1$, there is a NAC $\operatorname{Cone}_{\Gamma_i}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_i}) \leq i+1$.

For $i = 0$, we note that Γ_0 is the ball of radius 1 around ℓ in κ and can be written as $\Gamma_0 = \{\ell\} * \operatorname{lk}_\kappa(\ell)$. By Proposition 4.5, there is NAC $\operatorname{Cone}_{\Gamma_0}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_0}) \leq 1$.

Let $1 \leq i \leq n+1$ and assume there is a NAC $\operatorname{Cone}_{\Gamma_{i-1}}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_{i-1}}) \leq i$. We will apply Corollary 4.15 in order to bound the cone radius of Γ_i. We note that any two $U_1, U_2 \in Y_i \setminus Y_{i-1}, U_1 \neq U_2$ are of the same dimension and thus are not connected by an edge. Let $U \in Z_i \setminus Z_{i-1}$. We will show that $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ is a join of the vertex $\{U + \ell\}$ with the complex spanned by all the other vertices in $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$.

Note that $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ is a clique complex, thus it is enough to show that for every vertex U' in $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ that is not $\ell + U$, there is an edge connecting U' and $\ell + U$. Let U' be such vertex.

First, we will deal with the case were $\dim(U') < i$. In that case (using the fact that $\dim(U) = i$), U' being in $\operatorname{lk}_{\kappa_0}(U)$ implies that $U' \leq U$ and thus $U' \leq U + \ell$, i.e., U' is connected by an edge to $U + \ell$ as needed.

Second, assume that $\dim(U') > i$. From the definition of Z_{i-1}, it follows that $U' \in Z_0$, i.e., that $\ell \leq U'$. Also, U' is in $\operatorname{lk}_{\kappa_0}(U)$ and thus $U \leq U'$. It follows that $U + \ell \leq U'$, i.e., U' is connected by an edge to $U + \ell$ as needed.

By Proposition 4.5, for every $U \in Z_i \setminus Z_{i-1}$, there is 0-cone function $\operatorname{Cone}_{\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}) \leq 1$.

By Corollary 4.15, it follows that there is a NAC $\operatorname{Cone}_{\Gamma_i}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_i}) \leq i+1$ as needed. $\qquad\square$

Corollary B.4. *For every $n \in \mathbb{N}$ there is a constant $\mathcal{R}(n)$ such that for every $T_{\mathcal{E}}(V) \in \mathbf{C}_A$ with $\dim T_{\mathcal{E}}(V) = n$ there in a NAC $\operatorname{Cone}_{T_{\mathcal{E}}(V)}$ such that*

$$\operatorname{Rad}(\operatorname{Cone}_{T_{\mathcal{E}}(V)}) \leq \mathcal{R}(n).$$

In particular, if Δ is a building of type A_n ($n \geq 2$) over a field K with $|K| \geq 2^{n-1}$ and $a \in \Delta$ a simplex, then there exists a NAC $\operatorname{Cone}_{\Delta^0(a)}$ of $\Delta^0(a)$ such that

$$\operatorname{Rad}(\operatorname{Cone}_{\Delta^0(a)}) \leq \mathcal{R}(n-1).$$

Proof. The result follows from Theorem 5.5 together with Lemma B.3 and Fact B.2. In the notation of Theorem 5.5, we have $f(n) = n+2$ and $\ell_n = n+1$. $\qquad\square$

C   Details for type C_n

In this section, we fill in the details of the proof that the complex opposite the fundamental chamber in a spherical building of type C_n has a non-Abelian cone function with bounded cone radius. The class $\mathbf{C}$ that will cover the case of opposite complexes in buildings of type C_n will be the union of class $\mathbf{C}_A$ (see Definition A.6) and $\mathbf{C}_C$ (see Definition A.10).

Again, the argument will follow [OVB25, Chapter 6.3] closely.

Proposition C.1. *The class $\boldsymbol{C}$ satisfies the requirements of Theorem 5.5.*

Corollary C.2. *Let $\Delta = \operatorname{Flag} X(V)$ be a classical C_n building as described in Appendix A.2.3 (note that $\dim \Delta = n - 1$). Assume that K is infinite or $K = \mathbb{F}_q$ and*

- *$q \geq 2^{2n-2}$ if f is alternating and $Q = 0$,*

- *$q \geq 2^{4n-4}$ if $m = 2n$ and $\sigma \neq \operatorname{id}$,*

- *$q \geq 2^{2n-1}$ if $m = 2n + 1$ or $2n + 2$.*

In particular, if Δ is the building coming from the BN-pair of a Chevalley group of type B_n or C_n over the field $\mathbb{F}_q$ we require $q \geq 2^{2n-1}$.

Then for any $a \in \Delta$ there exists a NAC $\operatorname{Cone}_{\Delta^0(a)}$ of $\Delta^0(a)$ with

$$\operatorname{Rad}(\operatorname{Cone}_{\Delta^0(a)}) \leq \mathcal{R}(n-1)$$

where $\mathcal{R}(n-1)$ is as in Theorem 5.5 with $f(n) = 2n + 1$ and $\ell_n = 2n + 2$.

The main part of the proof of Proposition C.1 is covered by the following lemma.

Lemma C.3. *Let $(T_{\mathcal{E}}(V), \operatorname{Flag} X) \in \boldsymbol{C}_C$. Set $n = \dim \operatorname{Flag} X \geq 1$. By [Abr96], there exists a 1-dimensional subspace $\ell \in X_{\mathcal{E}}(V)$. Define $X_0 = \{U \in X \mid U \text{ satisfies } 1, 2 \text{ or } 3\}$ where*

  (i) $\ell \leq U$ and $U \pitchfork \mathcal{E}$;

  (ii) $\ell \nleq U \leq \ell^\perp$ and $U \pitchfork \mathcal{E} \cup (\mathcal{E} + \ell)$;

  (iii) $U \nleq \ell^\perp, \dim U > 1$ and $U \pitchfork \mathcal{E} \cup (\mathcal{E} \cap \ell^\perp) \cup (\mathcal{E} + \ell) \cup ((\mathcal{E} \cap \ell^\perp) + \ell)$.

Then $\kappa_0 := \operatorname{Flag} X_0$ has a NAC $\operatorname{Cone}_{\kappa_0}$ such that

$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq 2n + 3.$$

Proof. Let $(T_{\mathcal{E}}(V), \operatorname{Flag} X) \in \boldsymbol{C}_C$ as above. Thus $\dim \operatorname{Flag} X = n$ and $\operatorname{Flag} X$ is of type C_{n+1}. In particular, any totally isotropic subspace of V has dimension at most $n + 1$.

We will apply Corollary 4.15 inductively to two different filtrations of κ_0. The first one is defined as follows.

$$Y_0 = \{U \in X \text{ satisfying } 1.\}$$
$$Y_i = Y_{i-1} \cup \{U \in X \mid \dim U = i, U \text{ satisfies } (ii)\} = \{U \in X \mid \dim U \leq i, U \text{ satisfies } (ii)\} \cup Y_0$$
$$\text{for } 1 \leq i \leq n + 1.$$

We set $\Gamma_i = \operatorname{Flag} Y_i$.

The second filtration is given by

$$Z_0 = Y_{n+1}$$
$$Z_i = Z_{i-1} \cup \{U \in X \mid \dim U = n + 1 - (i - 1), U \text{ satisfies } (iii)\}$$
$$= Z_0 \cup \{U \in X \mid \dim U \geq n + 1 - (i - 1), U \text{ satisfies } (iii)\}$$

for $1 \leq i \leq n + 1$. We set $\zeta_i = \operatorname{Flag} Z_i$. We have that $\Gamma_{n+1} = \zeta_0$ and $\zeta_{n+1} = \kappa_0$.

Next, we want to show that for each Γ_i there exists a NAC Cone_{Γ_i} such that $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_i}) \leq i+1$. We do this by induction on i.

For $i = 0$ we have $\Gamma_0 = \mathrm{st}_\kappa(\ell) = \{\ell\} * \mathrm{lk}_\kappa(\ell)$. Hence by Section 4.2 there exists a NAC Cone_{Γ_0} with $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_0}) \leq 1$.

For the following step, fix $1 \leq i \leq n + 1$. We want to apply Corollary 4.15 to the following set-up. Here κ_0 corresponds to the complex called X in Corollary 4.15, Γ_{i-1} corresponds to X', and $Y_i \setminus Y_{i-1} = W_i$. We check that the conditions of Corollary 4.15 are satisfied.

For Condition (i) note that $W_i \cap \Gamma_{i-1} = \emptyset$ by definition. Condition (ii) holds since all subspaces in W_i have the same dimension, hence they are not connected in Δ and thus not in κ.

For Condition (iii) let $w \in W_i$. Then $\mathrm{lk}_{\kappa_0}(w) = \mathrm{Flag}\,\{u \in \kappa_0(0) \mid u < w \text{ or } w > u\}$. Furthermore, for $u \in Y_{i-1}$ we have $u + \ell \in X_0$, since $u \pitchfork \mathcal{E} + \ell \iff u + \ell \pitchfork \mathcal{E}$, see [Abr96]. Since $\ell \leq w + \ell$, we have $w + \ell \in Y_0 \subset Y_{i-1}$. Next, we want to show that $\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$ is a join of $\{w + \ell\}$ with the flag complex of all other vertices from $\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$. Since $\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$ is a clique complex, it suffices to check that every vertex different from $w + \ell$ is adjacent to $w + \ell$. Let $u \in (\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1})(0) \setminus \{w + \ell\}$. Then we differentiate two cases:

- Case 1: $\dim u \leq i - 1$: then $u \leq w \leq w + \ell$.

- Case 2: $\dim u > i$ then $u \in Y_0$, thus $w \leq u$ and $\ell \leq u$ thus $w + \ell \leq u$.

Hence any other vertex is connected to $w + \ell$ by an edge and thus

$$\Gamma_0 = \{w + \ell\} * \mathrm{Flag}\,((\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1})(0) \setminus \{w + \ell\})\,.$$

By Proposition 4.5 we have a 0-cone function $\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}}$ with $\mathrm{Rad}_0(\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}}) \leq 1$.

Thus we can apply Corollary 4.15 and get a NAC Cone_{Γ_i} satisfying $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_i}) \leq i + 1$.

We treat the second filtration similarly. We want to show that for every $0 \leq i \leq n + 1$ there is a NAC Cone_{ζ_i} such that $\mathrm{Rad}(\mathrm{Cone}_{\zeta_i}) \leq n + 2 + i$. We again proceed by induction on i.

For $i = 0$ we have that $\zeta_0 = \Gamma_{n+1}$ has cone function with radius $\leq n + 2$ by the above reasoning.

Now fix $1 \leq i \leq n + 1$. We want to use Corollary 4.15 with κ_0 corresponding to X, ζ_{i-1} to X', and $Z_i \setminus Z_{i-1} = W$. Then conditions (i) and (ii) of Corollary 4.15 are satisfied by the same argument as above.

For Condition (iii), fix $w \in W$ and consider $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1} = \mathrm{Flag}\,\{u \in Z_{i-1} \mid u < w \text{ or } u > w\}$. We show that every vertex (different from $w \cap \ell^\perp$) is connected to $w \cap \ell^\perp$ hence $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1} = \{w \cap \ell^\perp\} * \mathrm{Flag}\,\{u \in Z_{i-1} \setminus \{w \cap \ell^\perp\} \mid u < w \text{ or } u > w\}$. Thus by Proposition 4.5, $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$ has a 0-cone function with cone radius bounded by 1. Applying Corollary 4.15 we get a NAC Cone_{ζ_i} satisfying

$$\mathrm{Rad}(\mathrm{Cone}_{\zeta_i}) \leq \mathrm{Rad}(\mathrm{Cone}_{\zeta_{i-1}}) + 1 \leq n + 2 + i\,.$$

To show that every vertex (different from $w \cap \ell^\perp$) is connected to $w \cap \ell^\perp$, let $u \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}(0) = \{u \in Z_{i-1} \mid u < w \text{ or } u > w\}$. If $w \leq u$ then $w \cap \ell^\perp \leq u$ and thus $\{u, w \cap \ell^\perp\} \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$. If $u \leq w$, then $\dim u < \dim w = n + 1 - i + 1$ hence $u \in Z_0$. Therefore u satisfies either 1 or 2. But if it satisfies 1, then $\ell \leq u \leq w$, a contradiction to $w \in Z_i \setminus Z_{i-1}$. Hence u satisfies 2, and in particular $u \leq \ell^\perp$. Thus $u \leq w \cap \ell^\perp$ and hence $\{u, w \cap \ell^\perp\} \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$.

Since $\kappa_0 = \zeta_{n+1}$ we get the desired result. $\qquad\square$

Proof of Proposition C.1. We proof that every $(\kappa, \Delta) \in \mathbf{C}$ has a filtration satisfying the assumptions of Theorem 5.5 by induction on $n = \dim \Delta$.

Let $n = 0$. Hence Δ will be a building of type A_1 or C_1. By [Abr96] there exists at least on 1-dimensional subspace $\ell \in X_{\mathcal{E}}(V)$ and thus κ will be non-empty.

For the inductive step, fix $n \geq 1$ and set $f(n) = 2n + 3$. We distinguish between two cases:

If $(\kappa, \Delta) \in \mathbf{C}_A$, Appendix B gives the desired filtration with $f_A(n) = n + 2 \leq f(n)$ and length of the filtration being $n + 1$.

If $(\kappa, \Delta) \in \mathbf{C}_C$, we can use the notation as in Definition A.10, e.g. $(\kappa, \Delta) = (T_{\mathcal{E}}(V), \text{Flag } X)$. As in Lemma C.3 we fix a 1-dimensional subspace $\ell \in X_{\mathcal{E}}(V)$ and define $X_0 = \{U \in X \mid U$ satisfies $1, 2$ or $3\}$ (where 1,2,3 are as in the lemma) and $\kappa_0 = \text{Flag } X_0$. Hence, by Lemma C.3, there exists a NAC Cone_{κ_0} such that

$$\text{Rad}(\text{Cone}_{\kappa_0}) \leq 2n + 3 = f(n).$$

The remaining filtration is defined as in the Abelian case, see [OVB25, Proposition 6.7], and hence satisfies the assumptions of Theorem 5.5. □

D   Details for type D_n

In this section, we fill in the details of the proof that the complex opposite the fundamental chamber in a spherical building of type D_n has a non-Abelian cone function with bounded cone radius.

The argument will follow [OVB25, Chapter 6.4] closely which itself relies on the work in [Abr96, Section 7] and we also refer to that book for more details. We will define three subclasses $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ and first show that $\mathbf{C}_1, \mathbf{C}_2$ satisfy the assumptions of Theorem 5.18 to then conclude that $\mathbf{C}_D = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3$ satisfies them as well.

We fix n to be the Witt index of (V, Q, f) hence $\dim V = 2n$ and we are dealing with a D_n building, which is $n - 1$-dimensional. Recall $X = \{0 < U < V \mid U$ totally isotropic$\}$.

Definition D.1. [Abr96, Definition 12] Let $U, E < V$, $\dim U < n, \dim E = n$. Assume that E is not totally isotropic. Then

$$U @ E :\iff U^{\perp} \cap E \text{ is not totally isotropic.}$$

Set

$$\mathcal{U}_n = \mathcal{U}_n(V) = \{A < V \mid \dim A = n, A \notin X\}$$
$$\mathcal{M}(A) = \{M < A \mid \dim M = n - 1, M \in X\} \text{ for any } A \in \mathcal{U}_n.$$

Definition D.2. We define the class $\mathbf{C}_1$ to consist of complexes $\text{Flag } X_{\mathcal{E};\mathcal{F}}(U; V)$ where

- (V, Q, f) is thick pseudo-quadratic space of Witt index n, in particular $\dim V = 2n$;

- $U \in X$ with $\dim U = k \geq 2$;

- $\mathcal{E}$ a finite set of subspaces of U, set $e_j = |\mathcal{E}_j|, 1 \leq j \leq k - 1$ like before;

- Let $\mathcal{F} = \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_{k-1}$ be finite subsets of $\mathcal{U}_n(V)$ satisfying

  (i) $U \cap F \in \mathcal{E} \cup \{0\}$ for all $F \in \mathcal{F}$;
  (ii) $\mathcal{F}_j^{\perp} = \mathcal{F}_j$;
  (iii) $\dim(U \cap F) \leq k - 1 - j$ for all $F \in \mathcal{F}_j, 1 \leq j \leq k - 1$.

- Assume that $|K| \geq \sum_{j=1}^{k-1} \binom{k-2}{j-1} e_j + 2^{k-1} s$, where $s = |\mathcal{F}|$.

Set
$$X_{\mathcal{E};\mathcal{F}}(U;V) := \{0 < W < U \mid W \pitchfork_U \mathcal{E} \text{ and } W @_V \mathcal{F}_j \text{ for } \dim W = j\}.$$

Lemma D.3. *Let* $\kappa = \operatorname{Flag} X_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_1$ *with* $k = \dim U \geq 2$, *in particular* κ *has dimension* $k-2$. *Then there exists a 1-dimensional subspace* $\ell \in X_{\mathcal{E};\mathcal{F}}(U;V)$. *We set* $Y = X_{\mathcal{E};\mathcal{F}}(U;V)$ *and define*
$$Y_0 := \{A \in Y \mid A + \ell \in Y\}.$$

Then $\kappa_0 = \operatorname{Flag} Y_0$ *has a NAC* $\operatorname{Cone}_{\kappa_0}$ *with radius at most* k.

Proof. Set $B_0 := \{A \in Y \mid \ell \leq A\}$, $\beta_0 = \operatorname{Flag} B_0$. Then $\beta_0 = \{\ell\} * \operatorname{lk}_Y(\{\ell\})$ and hence it has a NAC with cone radius bounded by 1. Next, set $B_i = \{A \in Y_0 \mid \ell \leq A \text{ or } \dim A \leq i\}$, $\beta_i = \operatorname{Flag} B_i$ for $1 \leq i \leq k-1$. Then all subspaces in $B_i \setminus B_{i-1}$ have dimension i. Let $A \in B_i \setminus B_{i-1}$. Then $\operatorname{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A + \ell\} * \operatorname{Flag}\{W \in B_{i-1} \mid B \neq A + \ell, A < W \text{ or } W < A\}$ by the same argument as in Lemma B.3. Applying Corollary 4.15 inductively, we get a NAC $\operatorname{Cone}_{\beta_{k-1}} = \operatorname{Cone}_{\kappa_0}$ with
$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq k.$$

$\square$

Lemma D.4. *The class* $\mathbf{C}_1$ *satisfies the assumptions of Theorem 5.18. In particular, any complex in* $\mathbf{C}_1$ *has a NAC with cone radius depending only on the dimension of the complex.*

Proof. The result follows from Remark 5.19, Lemma D.3 and [OVB25, Lemma 6.17]. $\square$

Definition D.5. We define the class $\mathbf{C}_2$ to consist of complexes $\operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V)$ where

- (V, Q, f) is a thick pseudo-quadratic space of Witt index n, in particular $\dim V = 2n$;

- $U \in X$ with $\dim U = k \geq 2$;

- $\mathcal{E}$ a finite set of subspaces of U, set $e_j = |\mathcal{E}_j|, 1 \leq j \leq k-1$ like before;

- $\mathcal{F} \subset \mathcal{U}_n(V)$ with $|\mathcal{F}| = s < \infty$ such that

    (i) $U \cap \mathcal{F} \subseteq \mathcal{E} \cup \{0\}$,
    (ii) $U \cap F^\perp = 0$ and $\dim(U \cap F) \leq 1$ for all $F \in \mathcal{F}$.

- Assume $|K| \geq \sum_{j=1}^{k-1} \binom{k-2}{j-1} e_j + 2s$.

- Set
$$Z_{\mathcal{E};\mathcal{F}}(U;V) := \{0 < W < U \mid W \pitchfork_U \mathcal{E} \text{ and } W @_V \mathcal{F}\}.$$

Lemma D.6. *Let* $\kappa = \operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_2$, $\dim U = k \geq 2$, *in particular* $\dim \kappa = k-2$. *Set* $Y = Z_{\mathcal{E};\mathcal{F}}(U;V)$. *Then there exists* $H < U$ *with* $\dim H = k-1$ *and* $H \in Y$. *We define*
$$Y_0 := \{B \in Y \mid B \cap H \in Y\}.$$

Then $\kappa_0 = \operatorname{Flag} Y_0$ *is* $(k-2)$-*dimensional and has a NAC* $\operatorname{Cone}_{\kappa_0}$ *with*
$$\operatorname{Rad}(\operatorname{Cone}_{X_0}) \leq k.$$

Proof. The proof is similar to the proofs of Lemma B.3 and Lemma D.3, the difference being that we now look at the dual version, where we fix a hyperplane instead of a line.

Set $B_0 = \{A \in Y \mid A \leq H\}, \beta_0 = \text{Flag } B_0$. Then $\beta_0 = \{H\} * \text{lk}_{Y_0}(\{H\})$ hence it has a NAC with cone radius bounded by 1.

Next, set $B_i = \{A \in Y_0 \mid A \leq H \text{ or } \dim A \geq k - i\}, \beta_i = \text{Flag } B_i$ for $1 \leq i \leq k - 1$. Then the subspaces in $B_i \setminus B_{i-1}$ are all of dimension $k - i$. Let $A \in B_i \setminus B_{i-1}$, then $\text{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A \cap H\} * \text{Flag}\{W \in B_{i-1} \mid W \neq A \cap H, W < A \text{ or } A < W\}$. Hence it has a 0-cone function with cone radius bounded by 1. Thus we can apply Corollary 4.15 inductively and obtain a NAC $\text{Cone}_{B_{k-1}} = \text{Cone}_{\kappa_0}$ with

$$\text{Rad}(\text{Cone}_{\kappa_0}) \leq k.$$

$\square$

Lemma D.7. *The class* $\mathbf{C}_2$ *satisfies the assumptions of Theorem 5.18. In particular, any complex in* $\mathbf{C}_2$ *has a NAC with cone radius bounded depending only on its dimension.*

Proof. The result follows from Remark 5.19, Lemma D.6 and [OVB25, Lemma 6.20]. $\square$

Next, we will define a third class, which will be the one containing the relevant opposite complexes, see also Remark D.11.

Definition D.8. We define the class $\mathbf{C}_3$ to consist of complexes of the form $\text{Flag } Y_{\mathcal{E}}(V)$ where

- (V, Q, f) is a thick pseudo-quadratic space of Witt index n, in particular $\dim V = 2n$;

- $\mathcal{E} = \mathcal{E}^{\perp}$ is a finite set of subspaces of V, set $\mathcal{E}_j = \{E \in \mathcal{E} : \dim E = j\}, e_j = |\mathcal{E}_j|$;

- $|K| \geq 2 \sum_{j=1}^{2n-1} \binom{2n-2}{j-1} e_j$;

- $\widehat{\mathcal{E}}_n := \mathcal{E}_n \cap \mathcal{U}_n(V)$;

- $X = \{0 < U < V \mid U \text{ is totally isotropic}\}$.

Set

$$Y_{\mathcal{E}}(V) := \{U \in X \mid U \tilde{\pitchfork} \mathcal{E} \text{ and if } \dim U < n \text{ then } U @ \widehat{\mathcal{E}}_n\}.$$

Lemma D.9. *Let* $\kappa = \text{Flag } Y_{\mathcal{E}}(V) \in \mathbf{C}_3$ *with* $\dim V = 2n$. *Then there exists a 1-dimensional subspace* $\ell \in Y = Y_{\mathcal{E}}(V)$. *Set*

$$\mathcal{F} := \mathcal{E} \cup \left(\mathcal{E} \cap \ell^{\perp}\right) \cup (\mathcal{E} + \ell) \cup \left(\left(\mathcal{E} \cap \ell^{\perp}\right) + \ell\right) \quad and$$
$$\widehat{\mathcal{F}}_n := \mathcal{F}_n \cap \mathcal{U}_n(V).$$

Consider the following conditions

(i) $\ell \leq U, U \tilde{\pitchfork} \mathcal{E}$ *and if* $\dim U < n$ *then* $U @ \widehat{\mathcal{E}}_n$;

(ii) $\ell \nleq U \leq \ell^{\perp}$ *and* $\begin{cases} U \pitchfork \mathcal{E}, U @ \widehat{\mathcal{E}}_n & \text{if } \dim U = n - 1, \\ U \pitchfork \mathcal{E} \cup (\mathcal{E} + \ell), U @ \widehat{\mathcal{F}}_n & \text{if } \dim U \leq n - 2; \end{cases}$

(iii) $U \nleq \ell^{\perp}, \dim U > 1, U \tilde{\pitchfork} \mathcal{F}$ *and if* $\dim U < n$ *then* $U @ \widehat{\mathcal{F}}_n$.

We define $Y_0 := \{U \in X \mid U$ satisfies 1.,2., or 3. $\}$. Then $\kappa_0 = \mathrm{Flag}\, Y_0$ has a NAC function Cone_{κ_0} with

$$\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_0}) \leq 2n + 1, \ 0 \leq j \leq 1.$$

Proof. Let $\kappa = \mathrm{Flag}\, Y_{\mathcal{E}}(V) \in \mathbf{C}_3$ with $\dim V = 2n$. By Step 1. [Abr96, Proposition 14] there exists a 1-dimensional subspace $\ell \in Y$. Let Y_0 be defined as above. Recall that $X = \{0 < U < V \mid U$ is totally isotropic subspace$\}$. We define a filtration for Y_0 and we want to apply Corollary 4.15 inductively. We will use the fact stated in Step 3 of [Abr96, Proposition 14], that for every $U \in Y_0$, we have that $U \cap \ell^{\perp} \in Y_0$ and $(U \cap \ell^{\perp}) + \ell \in Y_0$.

Set

$$A_0 := \{U \in X \mid U \text{ satisfies 1.}\}$$
$$A_i := \{U \in X \mid U \text{ satisfies 1. or } (\dim U \leq i \text{ and } U \text{ satisfies 2.})\}, \ 1 \leq i \leq n;$$
$$\alpha_i := \mathrm{Flag}\, A_i, 0 \leq i \leq n.$$

Then $A_0 = \{\ell\} * \mathrm{Flag}\{U \in A_0 \mid U \neq \ell\}$ and hence has a NAC with cone radius bounded by 1. Clearly elements in $A_i \setminus A_{i-1}$ have the same dimension an are hence not adjacent. Let $U \in A_i \setminus A_{i-1}$. Then

$$\mathrm{lk}_{\alpha_i}(U) \cap \alpha_{i-1} = \mathrm{Flag}\{W \in X \mid (W \in A_0 \text{ and } U < W) \text{ or } (W \in A_{i-1} \text{ and } W < U)\}.$$

We want to show that each subspace in $\{W \in X \mid (W \in A_0 \text{ and } U < W) \text{ or } (W \in A_{i-1} \text{ and } W < U)\}$ is either contained in or contains $U + \ell$ (note that we have $U \cap \ell^{\perp} + \ell = U + \ell \in A_0 \subseteq A_{i-1}$). If $W \in A_0$ and $U < W$ then $\ell \leq W$ and hence $U + \ell \leq W$. If $W \in A_{i-1}$ and $W < U$ then $W < U + \ell$ as needed. The second part of the filtration is defined as follows. We set

$$B_i := \{U \in X \mid U \in A_n \text{ or } (U \text{ satisfies 3. and } \dim U \geq n - i + 1\},$$
$$\beta_i := \mathrm{Flag}\, B_i, 0 \leq i \leq n.$$

Then $A_n = B_0$. Let $U \in B_i \setminus B_{i-1}$. We want to show that $\mathrm{lk}_{\beta_i}(U) \cap \beta_{i-1}$ can be written as the join of $\{U \cap \ell^{\perp}\}$ and $\mathrm{Flag}\{W \in B_{i-1} \mid W \neq U \cap \ell^{\perp}, W < U \text{ or } U < W\}$. First, note that $U \cap \ell^{\perp} \in Y_0$ and hence $U \cap \ell^{\perp} \in A_n \subseteq B_{i-1}$. Let $W \in B_{i-1} \setminus \{U \cap \ell^{\perp}\}$. If $W < U$ then $\dim W < n - i$, hence $W \in A_n \setminus A_0$, in particular $W \leq \ell^{\perp}$. Hence $W = W \cap \ell^{\perp} \leq U \cap \ell^{\perp}$. If $U < W$ then $U \cap \ell^{\perp} \leq U \leq W$. Thus we get a filtration of κ_0 of length $2n$. By applying Corollary 4.15 at each step, we get a NAC Cone_{κ_0} with

$$\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_0}) \leq 2n + 1, 0 \leq j \leq 1.$$

$\square$

We now have all the necessary ingredients to show that the relevant class of simplicial complexes satisfies the conditions of Theorem 5.18. The prove will be similar to the previous proofs of similar results.

Theorem D.10. *The class $\mathbf{C}_D = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3$ satisfies the conditions of Theorem 5.18.*

Proof. Let $\kappa = \mathrm{Flag}\, Y \in \mathbf{C}_D$. If $\kappa \in \mathbf{C}_1 \cup \mathbf{C}_2$ then the result follows from Lemma D.4 and Lemma D.7.

Now assume $\kappa = \mathrm{Flag}\, Y_{\mathcal{E}}(V) \in \mathbf{C}_3$. In this case the proof follows closely the proof of [Abr96, Proposition 14]. First of all, notice that we again have a 1-dimensional subspace $\ell \in Y_{\mathcal{E}}(V) =: Y$

(see Step 1 [Abr96, Proposition 14]). Let $n = \frac{1}{2} \dim V = \dim \kappa + 1$. We will again prove the existence of a filtration by induction on n, but this time starting with $n = 2$.

Let $n = 2$, hence $\dim \kappa = 1$. Step 2. in [Abr96, Proposition 14] shows that each vertex $U \in Y$ can be connected to ℓ by a path of length at most 5. Thus these paths define a 0-cone function Cone_κ of κ with radius at most 5.

Now assume $n \geq 3$. We define $\kappa_0 = \mathrm{Flag}\, Y_0$ as in Lemma D.9. In particular, we know that there exists a NAC Cone_{κ_0} with $\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_0}) \leq 2n + 1 = f(n)$ for all $0 \leq j \leq 1$.

The fact that the rest of the filtration satisfies the conditions of Theorem 5.18 follows from the corresponfing proof in the Abelian case, see [OVB25, Theorem 6.23]. $\qquad\square$

The following remark from [Abr96] gives the connection between $X_\mathcal{E}(V)$ and $Y_\mathcal{E}(V)$.

Remark D.11. Let $\tau = \{E_1, \dots E_r\}$ be a simplex of $\tilde{\Delta} = \mathrm{Orifl}\, \tilde{X}$ and set $\mathcal{E} = \mathcal{E}(a) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$. Then $\mathcal{E} \cap \mathcal{U}_n = \emptyset$, $Y_\mathcal{E}(V) = X_\mathcal{E}(V)$, $e_n \leq 2, e_{n-1} = e_{n+1} = 0$ and $e_i \leq 1$ for all other $1 \leq i \leq 2n - 1$.

Corollary D.12. *Let $\tilde{\Delta}$ be a building of type D_n over the field K, and $a \in \tilde{\Delta}$ be a simplex. If $|K| \geq 2^{2n-1}$, then $\tilde{\Delta}^0(a)$ has a NAC with $\mathrm{Rad}_j(\mathrm{Cone}_{\tilde{\Delta}^0(a)}) \leq 2\mathcal{R}(n-1), 0 \leq j \leq 1$, where $\mathcal{R}(n)$ does not depend on K.*

Proof. By Theorem D.10, we have that $Y_{\mathcal{E}(a)}(V)$ has NAC with radius $\leq \mathcal{R}(n-1)$. By Remark D.11, we have $X_{\mathcal{E}(a)}(V) = Y_{\mathcal{E}(a)}(V)$. Thus Proposition 5.10 yields a NAC for $\tilde{\Delta}^0(a) = \tilde{T}_{\mathcal{E}(a)}(V)$ with cone radius bounded by $2\mathcal{R}(n-1)$. $\qquad\square$

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